Serie di Taylor di una funzione

$$f(x) = \sum_{n=0}^{+\infty} \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n$$

$$\sim \sum_{n=0}^{K} \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n$$

$$\sim f(x_0) + \frac{f'(x_0)(x-x_0)^1}{1!} + \frac{f''(x_0)(x-x_0)^2}{2!} + \dots + o\left((x-x_0)^n\right)$$

se $x_0 = 0 \Rightarrow$ formula di McLaurin

$$\boxed{f(x) \sim f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)x^3}{3!} + \dots + \frac{f^{(n)}(0)x^n}{n!} + o(x^n)}$$

Funzione esponenziale

$$\boxed{e^x \sim 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + o(x^n)}, \text{ per } x \to 0$$

$$e^x = \sum_{n=0}^{+\infty} \frac{1}{n!} x^n$$

$$e^{\alpha x} = \sum_{n=0}^{+\infty} \frac{\alpha^n}{n!} x^n$$

$$e^{\alpha x^2} = \sum_{n=0}^{+\infty} \frac{\alpha^n}{n!} x^{2n}$$

per, $x \to 0$

Funzione Iperboliche.

$$\sinh x \sim x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + o\left(x^{2n+2}\right), \text{ per } x \to 0$$

$$\cosh x \sim 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} + o\left(x^{2n+1}\right), \text{ per } x \to 0$$

$$(\sinh x)' = \cosh x$$

$$(\cosh x)' = \sinh x$$

$$\sinh(0) = 0$$

$$\cosh(0) = 1$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Funzioni Trigonometriche

$$\sin x \sim x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o\left(x^{2n+2}\right), \text{ per } x \to 0$$

$$\cos x \sim 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + o\left(x^{2n+1}\right), \text{ per } x \to 0$$

Le funzioni $(1+x)^\alpha$

$$(1+x)^\alpha \sim 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \dots + \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}x^n + o(x^n)$$

$$\sim 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \dots + \binom{\alpha}{n}x^n + o(x^n)$$

Coefficiente Binomiale:

$$\binom{\alpha}{n} \overset{\text{def.}}{:=} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$$

- $\frac{1}{1+x} = (1+x)^{-1} \sim 1 - 1x + \frac{(-1)(-1-1)}{2}x^2 + \dots + \binom{-1}{n}x^n + o(x^n)$

  $\sim 1 - x + x^2 - x^3 + x^4 + \dots + (-1)^n x^n + o(x^n)$

- $\sqrt{1+x} = (1+x)^{+1/2} \sim 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots + \binom{1/2}{n}x^n + o(x^n)$

- $\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} \sim 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5}{16}x^3 + \dots + \binom{-1/2}{n}x^n + o(x^n)$

Proprieta' del Resto (simbolo di Landau "o")!

I) $o(x^n) \pm o(x^n) = o(x^n)$, per $x \to 0$

II) $o(x^n) \pm o(x^m) = o(x^p)$, con $p = \min(n, m)$, per $x \to 0$

III) $o(kx^n) = o(x^n)$, $\forall k \in \mathbb{R} \setminus \{0\} = \forall \{k : (k \in \mathbb{R}) \wedge (k \notin \{0\})\}$, per $x \to 0$

esclude il set dei zeri!

IV) $x^n \cdot o(x^m) = o(x^{n+m})$, per $x \to 0$

V) $o(x^n) \cdot o(x^m) = o(x^{n+m})$, per $x \to 0$

VI) $[o(x^n)]^k = o(x^{kn})$, per $x \to 0$

VII) $o(x^n) = x^n o(1)$, per $x \to 0$

VIII) $f = o(x^n) \wedge f \simeq g \Rightarrow g = o(x^n)$, per $x \to 0$

Sviluppo di $f(\alpha x)$

$$\begin{cases} f(x) = P_n(x) + o(x^n) \\ f(\alpha x) = P_n(\alpha x) + o((\alpha x)^n) = P_n(\alpha x) + o(x^n) \end{cases}$$

e.g.

(i)
$$\begin{cases} e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + o(x^n) \\ e^{3x} = 1 + (3x) + \frac{(3x)^2}{2!} + \dots + \frac{(3x)^n}{n!} + o(x^n) = 1 + 3x + \frac{9}{2}x^2 + \dots + \frac{3^n x^n}{n!} + o(x^n) \end{cases}$$

(ii)
$$\begin{cases} \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \dots + (-1)^n x^n + o(x^n) \\ \frac{1}{1-x} \overset{\boxed{x} \to \boxed{-x}}{=} 1 - (-x) + (-x)^2 - (-x)^3 + (-x)^4 + \dots + (-1)^n (-x)^n + o((-x)^n) \end{cases}$$

$$= 1 + x + x^2 + x^3 + x^4 + \dots + x^n + o(x^n)$$

Sviluppo della somma $f(x) \pm g(x)$

$$\begin{cases} f(x) = P_n(x) + o(x^n) \\ g(x) = Q_n(x) + o(x^n) \end{cases}$$

$$f(x) + g(x) = [P_n(x) + o(x^n)] + [Q_n(x) + o(x^n)] = [P_n(x) + Q_n(x)] + [o(x^n) + o(x^n)]$$

$$= P_n(x) + Q_n(x) + o(x^n)$$

e.g. (i) $f(x) = \sin x - \sinh x \longrightarrow \begin{cases} \sin x = x - \frac{x^3}{3!} + o(x^3) \\ \sinh x = x + \frac{x^3}{3!} + o(x^3) \end{cases}$

$$f(x) = \left[\cancel{x} - \frac{x^3}{3!} + o(x^3)\right] - \left[\cancel{x} + \frac{x^3}{3!} + o(x^3)\right]$$

$$= \frac{-2}{3!}x^3 + o(x^3) = -\frac{2}{3!}x^3 + o(x^3) = -\frac{2}{6}x^3 + o(x^3) = -\frac{1}{3}x^3 + o(x^3)$$

⊗ **Sviluppo del Prodotto** $\boxed{f(x)\cdot g(x)}$:

$$f(x)\cdot g(x) = \left[P_n(x)+o(x^n)\right]\cdot\left[Q_n(x)+o(x^n)\right]$$

$$= P_n(x)\cdot Q_n(x) + P_n(x)\cdot o(x^n) + Q_n(x)\cdot o(x^n) + o(x^n)\cdot o(x^n)$$

$$= P_n(x)\cdot Q_n(x) + o(x^n) + o(x^n) + o(x^n)$$

$$= P_n(x)\cdot Q_n(x) + o(x^n)$$

e.g.

$$f(x) = e^x \sin x \longrightarrow \begin{cases} \sin x = x - \frac{x^3}{3!} + o(x^4) \\ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4) \end{cases}$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)\left(x - \frac{x^3}{3!}\right) + o(x^4)$$

perché x^n con $n>4$

$$= x + x^2 + \frac{x^3}{2!} + \cancel{\frac{x^4}{3!}} + \cancel{\frac{x^5}{4!}} - \frac{x^3}{3!} - \cancel{\frac{x^4}{3!}} - \cancel{\frac{x^5}{2!3!}} - \cancel{\frac{x^6}{3!3!}} - \cancel{\frac{x^7}{3!4!}} + o(x^4)$$

$$= x + x^2 + \frac{x^3}{2} - \frac{x^3}{6} = x + x^2 + \frac{3x^3 - 1x^3}{6} = x + x^2 + \frac{x^3}{3} + o(x^4)$$

ok!

Sviluppo del Quoziente $\boxed{\dfrac{f(x)}{g(x)}}$:

$$h(x) = \frac{f(x)}{g(x)} \Longrightarrow \begin{cases} h(x)\cdot g(x) = f(x) \\ h(x) \overset{def.}{:=} \underbrace{S_n(x)}_{\text{e' un polinomio a coefficienti costanti trovati con il confronto!}} + o(x^n) \end{cases}$$

$$= c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + o(x^n)$$

e.g. $h(x) = \dfrac{\sin x}{\cos x} \Longrightarrow h(x)\cdot\cos x = \sin x.$; $\begin{cases} \sin x = x - \frac{x^3}{3!} + \frac{x^5}{120} + o(x^5) \\ \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) \end{cases}$

$$\left[c_0 + c_1 x + c_2 x^2 + \dots + c_5 x^5 + o(x^5)\right]\left[1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)\right] = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$$

$$c_0 + c_1 x + \left(c_2 + \frac{c_0}{2}\right)x^2 + \left(c_3 - \frac{c_1}{2}\right)x^3 + \left(c_4 + \frac{c_0}{24} - \frac{c_2}{2}\right)x^4 + \left(c_5 - \frac{c_3}{2} + \frac{c_1}{24}\right)x^5 + o(x^5) = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$$

confronto i coef. delle potenze di x dello stesso grado.

$$\begin{cases} c_0 = 0 \\ c_1 = 1 \\ c_3 - \frac{c_1}{2} = -1/6 \end{cases} \quad \begin{cases} c_4 + \frac{c_0}{24} - \frac{c_2}{2} = 0 \\ c_5 - \frac{c_3}{2} + \frac{c_1}{24} = 1/120 \end{cases} \Rightarrow \begin{cases} c_0 = 0 \\ c_1 = 1 \\ c_2 = 0 \\ c_3 = 1/3 \end{cases} \quad \begin{cases} c_4 = 0 \\ c_5 = 2/15 \end{cases}$$

$$\Rightarrow \boxed{\tan x = x + x^3/3 + 2/15\, x^5 + o(x^5)}$$ ok!

e.g. $h(x) = \dfrac{1}{\cos x} \implies [h(x)] \cdot [\cos x] = [1]$

$$\left[c_0 + c_2 x^2 + c_4 x^4 + o(x^4)\right] \cdot \left[1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right] = 1$$

$$c_0 + \left(c_2 - \frac{1}{2} c_0\right) x^2 + \left(c_4 - \frac{1}{2} c_2 + \frac{1}{24} c_0\right) x^4 + o(x^4) = 1$$

$$\begin{cases} c_0 = 1 \\ c_2 - \frac{1}{2} c_0 = 0 \\ c_4 - \frac{1}{2} c_2 + \frac{1}{24} c_0 = 0 \end{cases} \implies \begin{cases} c_0 = 1 \\ c_2 = 1/2 \\ c_4 = 5/24 \end{cases}$$

$$\implies \boxed{\frac{1}{\cos x} = 1 + \frac{1}{2} x^2 + \frac{5}{24} x^4 + o(x^4)}$$

Sviluppo della funzione composta $\boxed{g \circ f} = g(f(x))$

e.g. $h(x) = e^{\sin x} \longrightarrow \begin{cases} f(x) = \sin x = x - \frac{x^3}{6} + o(x^3) \\ g(y) = e^y = 1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + o(y^3) \end{cases}$

$$h(x) = g(f(x)) = 1 + \left(x - \frac{x^3}{6} + o(x^3)\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^3)\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{6} + o(x^3)\right)^3 + o\left[\left(x - \frac{x^3}{6} + o(x^3)\right)^3\right]$$

$$= 1 + x - \cancel{\frac{x^3}{6}} + o(x^3) + \frac{1}{2}x^2 + \cancel{\frac{1}{6}x^3} + o(x^3) = 1 + x + \frac{1}{2}x^2 + o(x^3)$$

e.g. $h(x) = e^{\frac{x}{1-x}} = g(f(x))$

$$f(x) = \frac{x}{1-x} = x \cdot \left(\frac{1}{1-x}\right) = x\left(1 + x + x^2 + x^3 + o(x^3)\right) = x + x^2 + x^3 + o(x^3)$$

$$g(y) = e^y = 1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + o(y^3)$$

NB!

$$\frac{1}{2}\left[x^2 + x^4 + 2x \cdot x^2\right] = \frac{1}{2}\left[x^2 + \cancel{x^4}^{\,0} + 2x^3\right]$$

$$h(x) = g(f(x)) = 1 + \left(x + x^2 + x^3 + o(x^3)\right) + \underbrace{\frac{1}{2}\left(x + x^2 + x^3 + o(x^3)\right)^2}_{} + \frac{1}{6}\left(x + x^2 + x^3 + o(x^3)\right)^3 + o(x^3)$$

$+x^3$ NB!

$$= 1 + x + x^2 + \underline{x^3} + o(x^3) + \frac{1}{2}x^2 + o(x^3) + \frac{1}{\underline{6}}x^3 + o(x^3)$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{13}{6}x^3 + o(x^3)$$

OK!

Serie di Taylor di una funzione

$$f(x) = \sum_{n=0}^{+\infty} \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n$$

$$\sim \sum_{n=0}^{K} \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n$$

$$\sim f(x_0) + \frac{f'(x_0)(x-x_0)^1}{1!} + \frac{f''(x_0)(x-x_0)^2}{2!} + \dots + o((x-x_0)^n)$$

se $x_0 = 0 \Rightarrow$ formula di McLaurin

$$\boxed{f(x) \sim f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)x^3}{3!} + \dots + \frac{f^{(n)}(0)x^n}{n!} + o(x^n)}$$