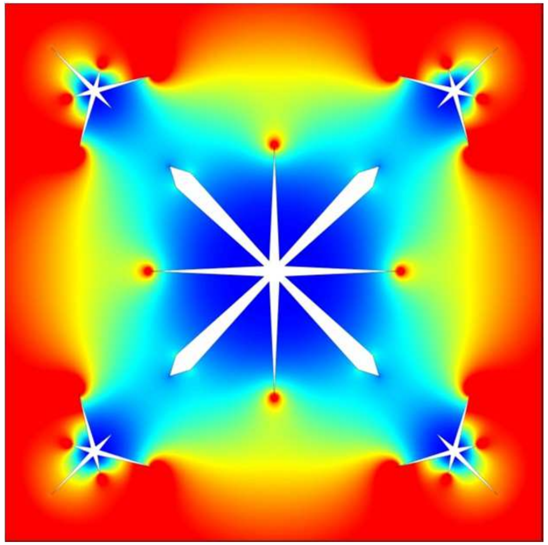
Trento, April 21, 2016

Summer Shahzad

Stress Singularities, Annihilations and Invisibilities Induced by Polygonal Inclusions in Linear Elasticity



Tutors: Davide Bigoni - Francesco Dal Corso

Department of Civil, Environmental and Mechanical Engineering
University of Trento - Italy

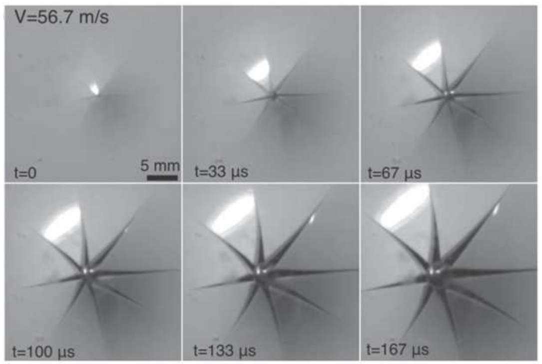
MOTIVATION

The <u>knowledge of the stress state</u> is essential in engineering design of <u>safe structures</u>

Structural failure of a lap joint in the fuselage skin due to metal fatigue cracks (Boing 737, April 1,2011)

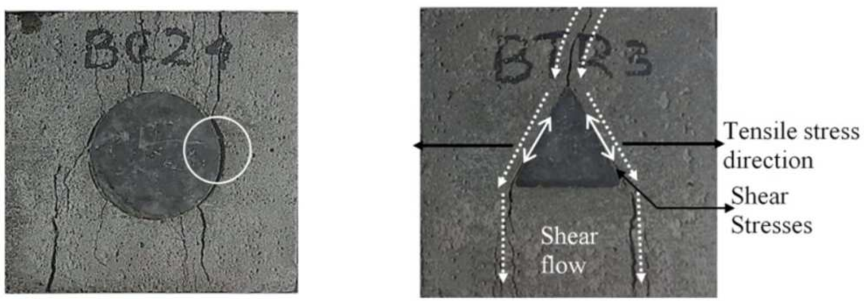
Two Critical conditions in composites materials are encountered in the presence of:

Soft phases (e.g. voids and defects)



Star cracks pattern on impacted PMMA plates (Ay Vandenberghe et al., 2013)

Stiff phases (e.g. rigid inclusions)



Cracking of mortar specimens containing cylindrical and triangular inclusion (Ay Lie Han et al., 2014)

MAIN RESEARCH GOALS

- **Rigid inclusion model:**
  Can the stress singular fields around stiff inclusions (predicted by linear elastic solution) be generated in reality?
- **Inclusion invisibility:**
  Can the presence of an inclusion leave the stress field unperturbed?
- **Stress annihilators:**
  Can the presence of an inclusion provide zero stress values at the inclusion vertexes?
- **Stress reducers:**
  Can the presence of an inclusion provide a finite stress smaller than that when the inclusion is absent?

PLANE ELASTICITY

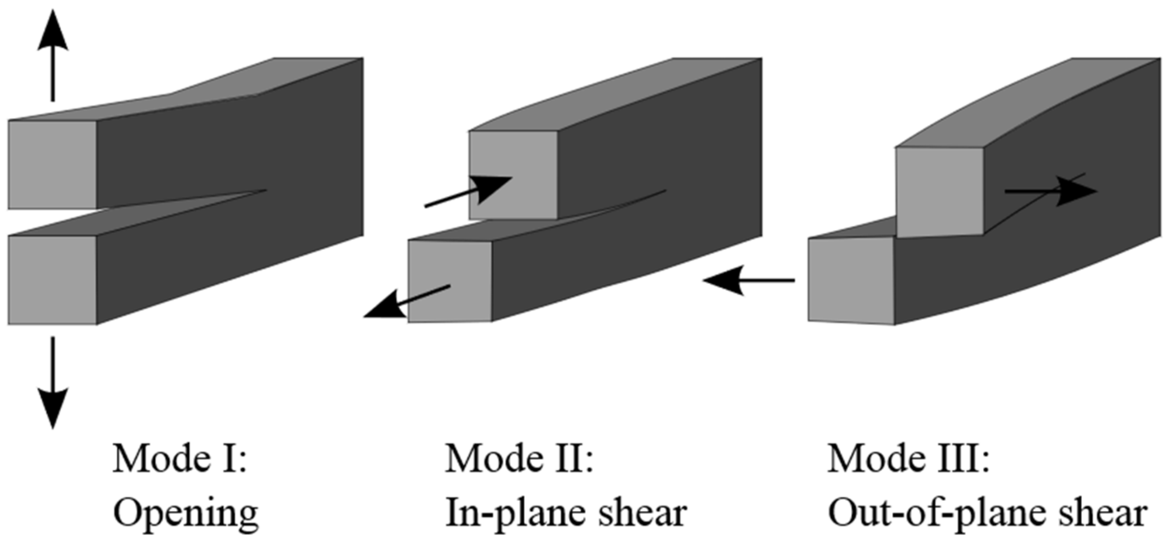
IN-PLANE ELASTICITY

$$\nabla^4 \Phi(x_1, x_2) = 0$$

OUT-OF-PLANE ELASTICITY

$$\nabla^2 u_3 \left(x_1, x_2\right) = 0$$

Fracture Modes

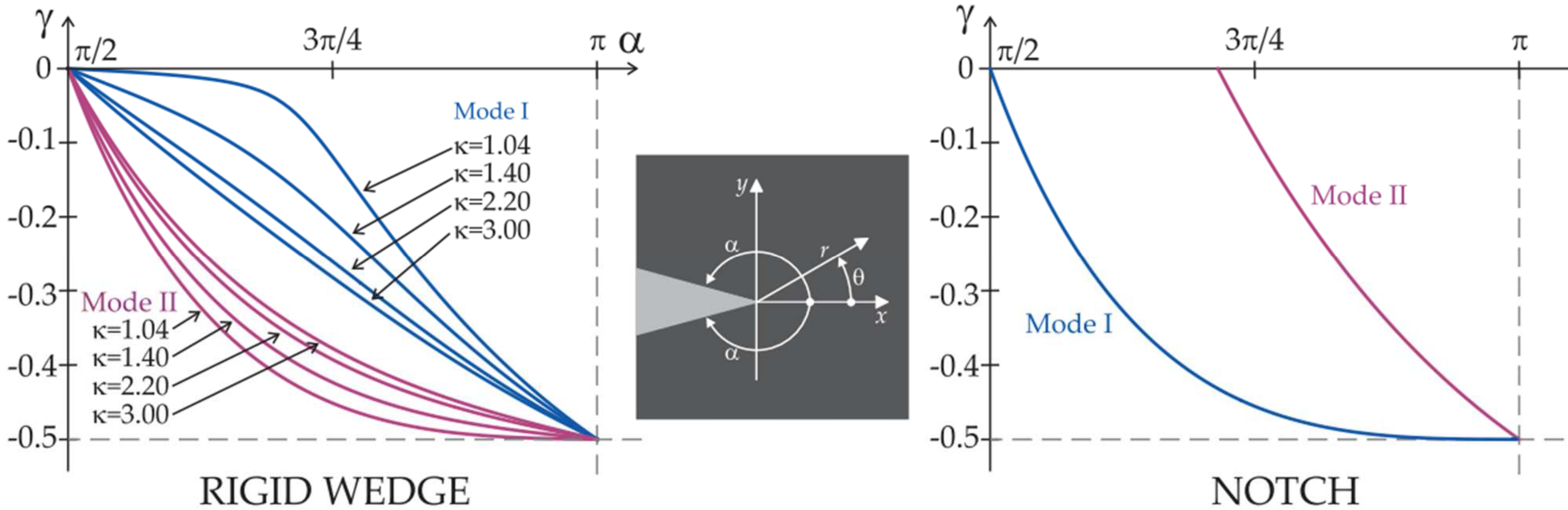


ASYMPTOTIC APPROACH

IN-PLANE ASYMPTOTICS

In-plane Modes

$$\{\sigma_{\alpha\beta}, \varepsilon_{\alpha\beta}\} \sim r^{\gamma}$$



For the rigid wedge, similarly to the notch problem:

- the singularity appears for $\alpha > \frac{\pi}{2}$;
- a square root singularity ($\sigma \sim \frac{1}{\sqrt{r}}$) appears for both mode I and mode II when α tends to π;

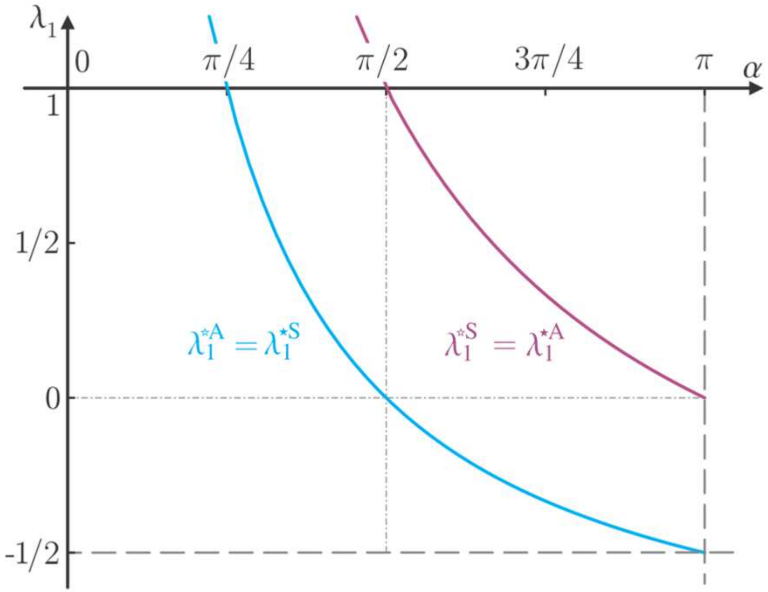
while, differently from the notch problem:

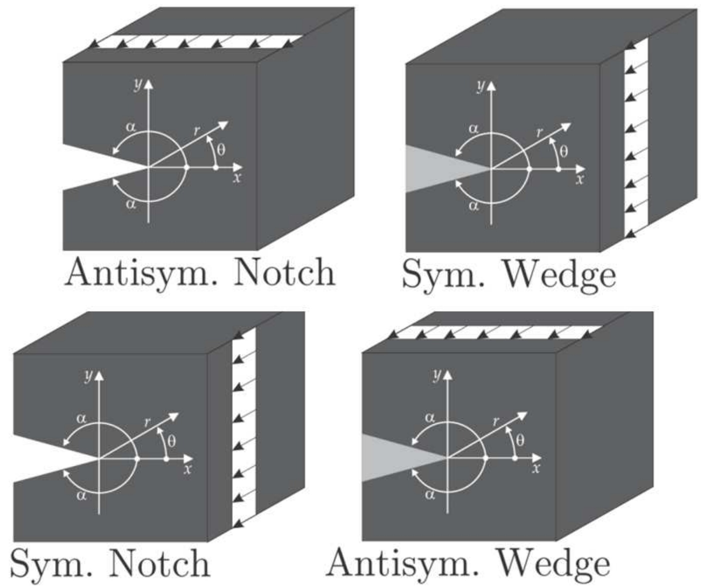
- the singularity depends on Poisson's ratio;
- the singularity is stronger in mode II than in mode I.

$$\kappa = \begin{cases} 3 - 4\nu, & \text{for plane strain,} \\ \dfrac{3-\nu}{1+\nu}, & \text{for plane stress,} \end{cases}$$

OUT-OF-PLANE ASYMPTOTICS

Out-of-plane Mode





For antisymmetric notch/symmetric wedge

- the singularity appears for $\alpha > \frac{\pi}{2}$;
- a square root singularity ($\sigma \sim \frac{1}{\sqrt{r}}$) appears when α tends to π;
- non singular leading order term appears for $\alpha < \frac{\pi}{2}$;
- a constant term appears when $\alpha = \frac{\pi}{2}$, commonly known as S-stress ;

For symmetric notch/antisymmetric wedge

- non singular leading order term appears for $\alpha < \pi$;
- a constant term appears when $\alpha = \pi$, namely S-stress ;

METHOD OF ANALYTIC FUNCTIONS

WHAT ABOUT THE FULL-FIELD SOLUTION

SCHWARZ-CHRISTOFFEL CONFORMAL MAPPING [1]

For a generic **N-sided polygon**

Exterior angles at the polygonal vertices

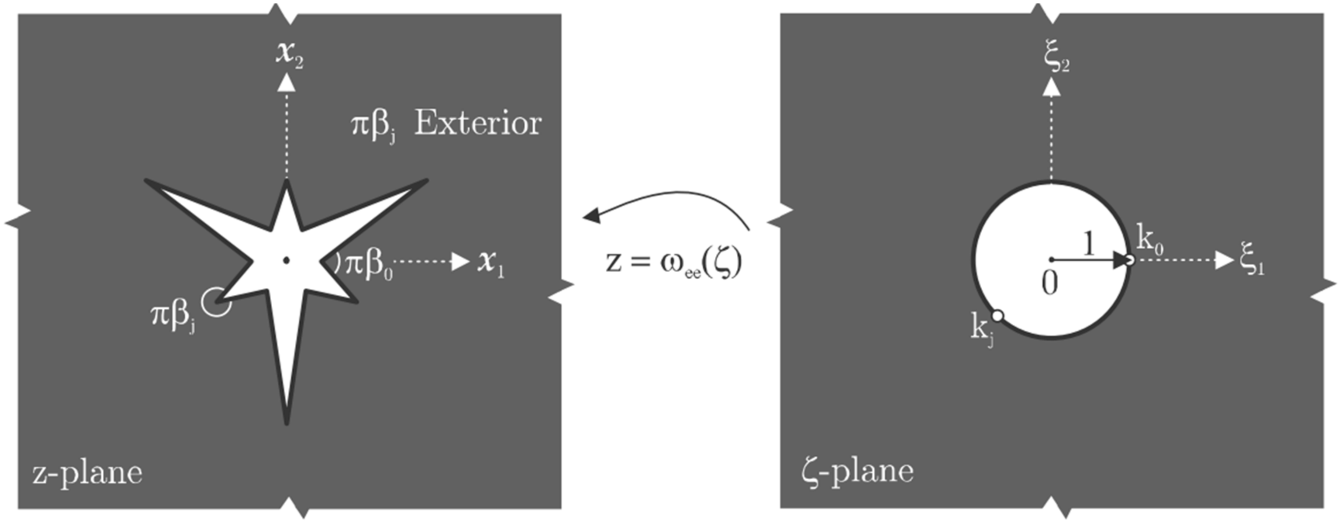
Scaling and rotation

$$\omega_{ee}(\zeta) = A \int_{1}^{\zeta} \left[\frac{1}{\sigma^2} \prod_{j=0}^{N-1} (\sigma - k_j)^{\beta_j - 1} \right] \mathrm{d}\sigma + B$$

Translation

Pre-images on the unit circle

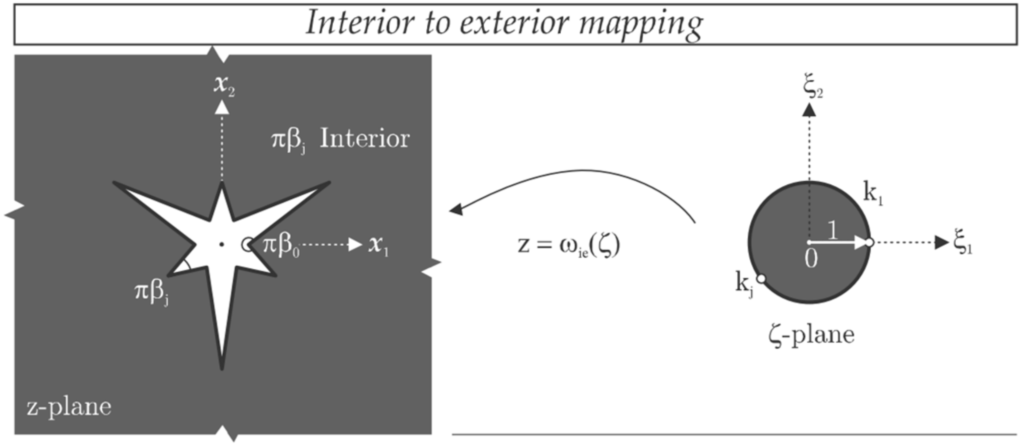
Exterior to Exterior mapping



SCHWARZ-CHRISTOFFEL CONFORMAL MAPPING [2]

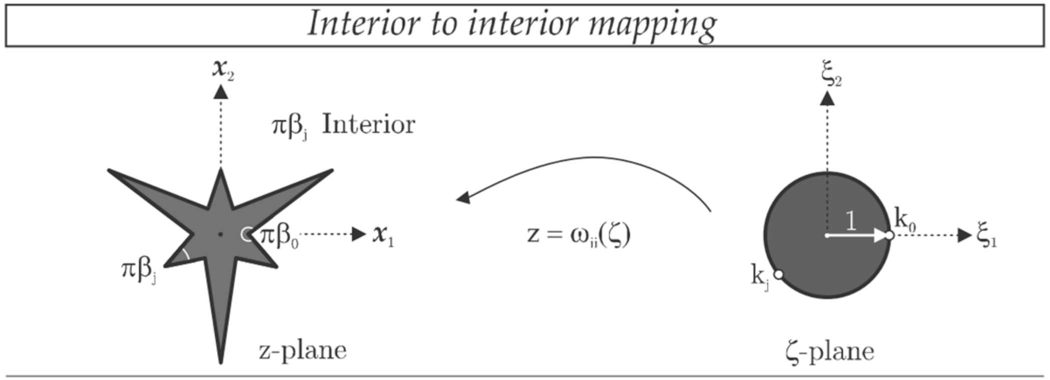
Interior to Exterior

$$\omega_{ie}(\zeta) = A\int_1^{\zeta}\left[\frac{1}{\sigma^2}\prod_{j=0}^{N-1}\left(1-\frac{\sigma}{k_j}\right)^{1-\beta_j}\right]d\sigma + B$$



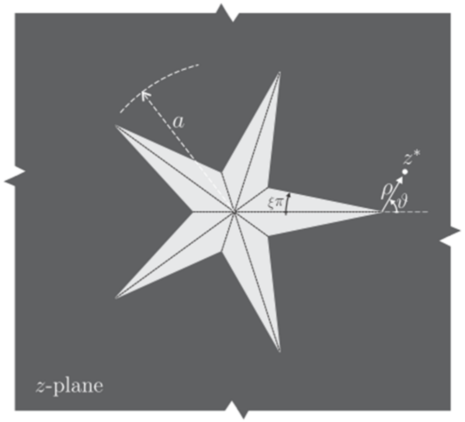
Interior to Interior

$$\omega_{ii}(\zeta) = A\int_1^{\zeta}\left[\prod_{j=0}^{N-1}\left(1-\frac{\sigma}{k_j}\right)^{\beta_j-1}\right]d\sigma + B$$



SCHWARZ-CHRISTOFFEL CONFORMAL MAPPING [3]

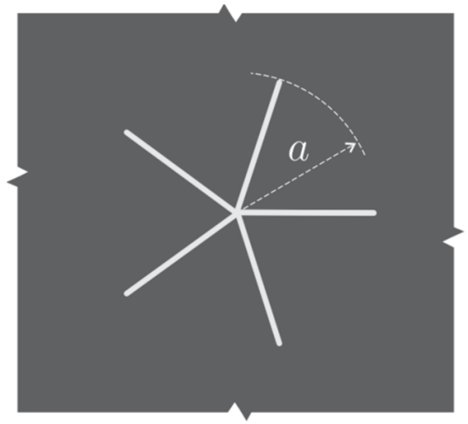
Non-intersecting Isotoxal star-shaped polygon



$$\omega'(\zeta,\xi,n) = a\Omega(n,\xi)\frac{(\zeta^n-1)^{1-2\xi}\,(\zeta^n+1)^{2\left(\xi+\frac{1}{n}\right)-1}}{\zeta^2}$$

$$\omega(\zeta,\xi,n) = a\Omega(n,\xi)\,\zeta\,F_1\left(-\frac{1}{n};2\xi-1,1-2\xi-\frac{2}{n};1-\frac{1}{n};\frac{1}{\zeta^n},-\frac{1}{\zeta^n}\right)$$

Star-shaped crack/stiffener



$$\omega'(\zeta,n) = a\Omega(n)\left(1-\frac{1}{\zeta^n}\right)\left(1+\frac{1}{\zeta^n}\right)^{\frac{2-n}{n}}$$

$$\omega(\zeta,n) = \frac{a}{\sqrt[n]{4}}\,\zeta\left(1+\frac{1}{\zeta^n}\right)^{\frac{2}{n}}$$

Laurent series

$$\omega(\zeta,\xi,n) = a\Omega(n,\xi)\sum_{j=0}^{\infty} d_{1-jn}(\xi)\,\zeta^{1-jn}$$

$$d_{1-jn}(\xi) = \frac{1}{1-jn}\sum_{k=0}^{j}\frac{(-1)^{j-k}}{k!\,(j-k)!}\frac{\Gamma\left(1-\frac{2}{n}-2\xi+j-k\right)\Gamma\left(-1+2\xi+k\right)}{\Gamma\left(1-\frac{2}{n}-2\xi\right)\Gamma\left(-1+2\xi\right)}$$

$$\Omega(n,\xi) = \frac{1}{\sqrt[n]{4}}\frac{\Gamma\left(1-\frac{1}{n}-\xi\right)}{\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(1-\xi\right)} \quad \in \left[\frac{1}{2},1\right)$$

$$d_{1-jn} = \frac{1}{j!}\prod_{k=0}^{j-1}\left(\frac{2}{n}-k\right)$$

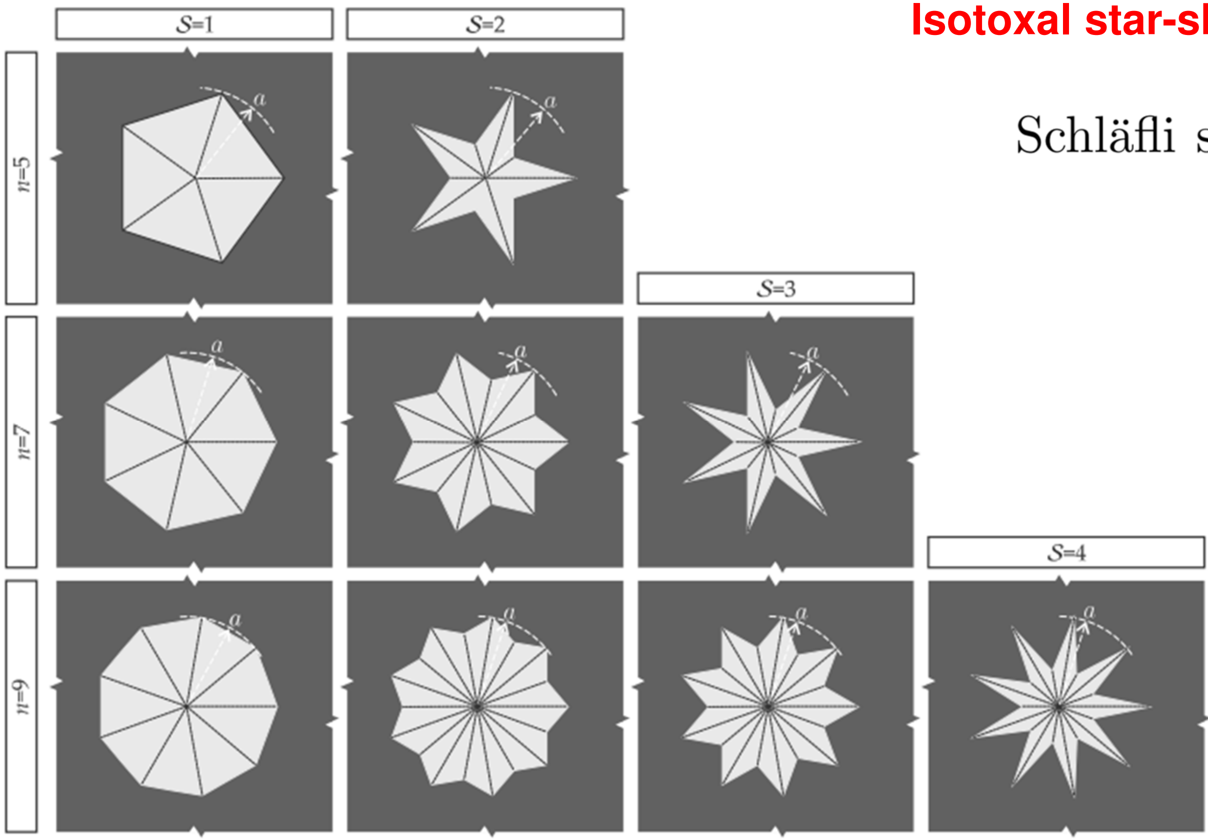
$$\Omega(n) = \frac{1}{\sqrt[n]{4}} \quad \in \left[\frac{1}{2},1\right)$$

SCHWARZ-CHRISTOFFEL CONFORMAL MAPPING [4]

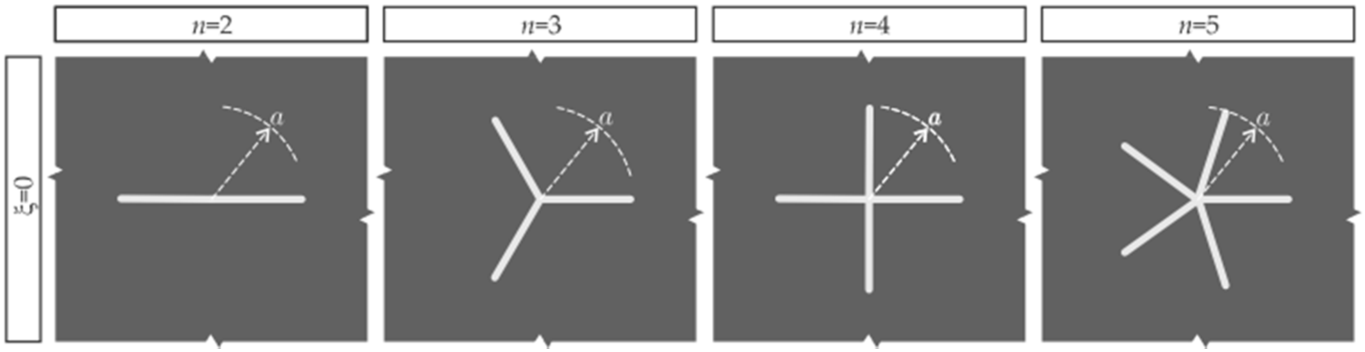
Isotoxal star-shaped polygon

Schläfli symbol $|n/\mathcal{S}|$

$$\mathcal{S} < n/2$$



Special cases of Star-shaped crack/stiffener



IN-PLANE PROBLEM [1]

(Kolosov-Muskhelishvili)

Is governed by

$$\nabla^4 \Phi(x_1, x_2) = 0$$

Airy Stress Function in complex plane

Unknown Complex Potentials

$$\Phi\left(z, \overline{z}\right) = \mathrm{Re}\left[\overline{z}\,\varphi(z) + \int \psi(z)\,dz\right]$$

Boundary conditions for void (null traction) and rigid inclusions

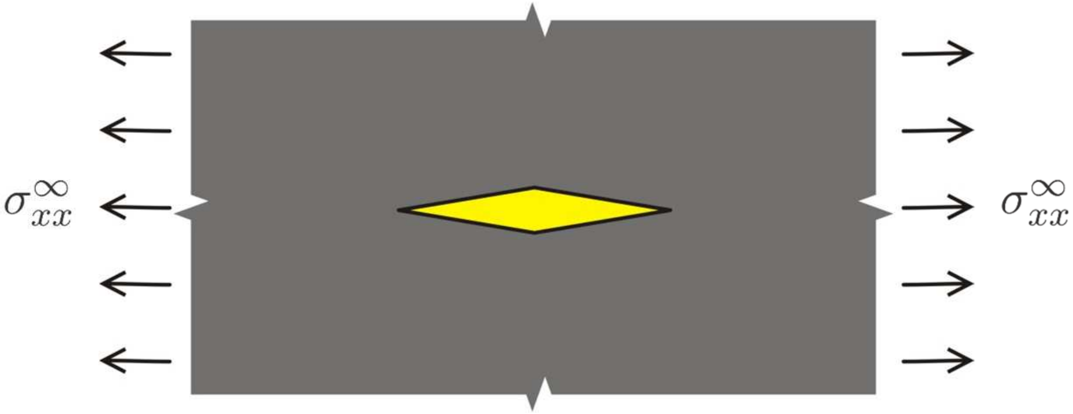
$$\Theta\varphi(z) + \chi z\,\overline{\varphi'(z)} + \chi\overline{\psi(z)} = (1-\chi)\,\mathrm{i}\,\mu\epsilon z$$

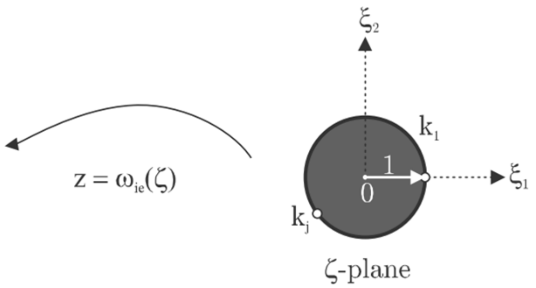
$$(\Theta;\,\chi) = \begin{cases} (1;\,1) & \text{for} \quad \text{void,} \\ (\kappa;\,-1) & \text{for} \quad \text{rigid inclusion,} \end{cases}$$

Rotation of the inclusion induced by External loadings

Perfectly bonded infinitely rigid inclusion embedded in an infinite plane

$\epsilon = 0$ due to the symmetry





For example, for mode I these potentials can be written as the sum of unperturbed and perturbed fields

$$\varphi(\zeta) = \frac{\sigma_{xx}^{\infty}}{4}\omega(\zeta) + \varphi^{(p)}(\zeta), \qquad \varphi^{(p)}(\zeta) = \sum_{j=1}^{\infty} a_j \zeta^j$$

$$\psi(\zeta) = -\frac{\sigma_{xx}^{\infty}}{2}\omega(\zeta) + \psi^{(p)}(\zeta), \qquad \psi^{(p)}(\zeta) = \sum_{j=1}^{\infty} b_j \zeta^j$$

IN-PLANE PROBLEM [3]

B.C.

$$\kappa\varphi^{(p)}(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi^{(p)\prime}(\zeta)} - \overline{\psi^{(p)}(\zeta)} = \frac{\sigma_{xx}^{\infty}}{2}\left(\frac{1-\kappa}{2}\omega(\zeta) - \overline{\omega(\zeta)}\right), \qquad \text{for } \zeta = e^{\mathrm{i}\theta},\ \theta \in [0, 2\pi]$$

Cauchy integral formula

$$f(\sigma) = \frac{f^*(\sigma)}{(\sigma - \sigma_j)^n}$$

Order of the pole

$$\frac{1}{2\pi\,\mathrm{i}}\oint_{|\gamma|=1}\frac{f(\sigma)}{\sigma-\zeta}d\sigma = f(\zeta) - \sum_{j=0}^{k} g_j(\zeta), \quad \text{if } \zeta \in D^+ \qquad g_j(\zeta) = \lim_{\sigma\to\sigma_j}\left[\sum_{l=0}^{n-1}\frac{1}{l!}\frac{f^{*(l)}(\sigma)}{(\zeta-\sigma_j)^{n-l}}\right]$$

j-th Pole

e.g.

$$f(\sigma) = \frac{\sigma^3+1}{\sigma(\sigma-\frac{1}{2})^3} \qquad \begin{cases} g_1(\zeta) = -\dfrac{8}{\zeta}, \\ g_2(\zeta) = \dfrac{9}{4\left(\zeta-\frac{1}{2}\right)^3} - \dfrac{3}{\left(\zeta-\frac{1}{2}\right)^2} + \dfrac{9}{\zeta-\frac{1}{2}} \end{cases}$$

IN-PLANE PROBLEM [4]

Perturbed Potentials

$$\varphi^{(p)}(\zeta) = \left(-0.1628 + 0.0071\zeta^2 + 0.0001\zeta^4 + 0.0009\zeta^6 + 0.0001\zeta^8 + 0.0003\zeta^{10} + 0.0001\zeta^{12} + 0.0002\zeta^{14}\right) R\,\sigma_{xx}^{\infty}\,\zeta,$$

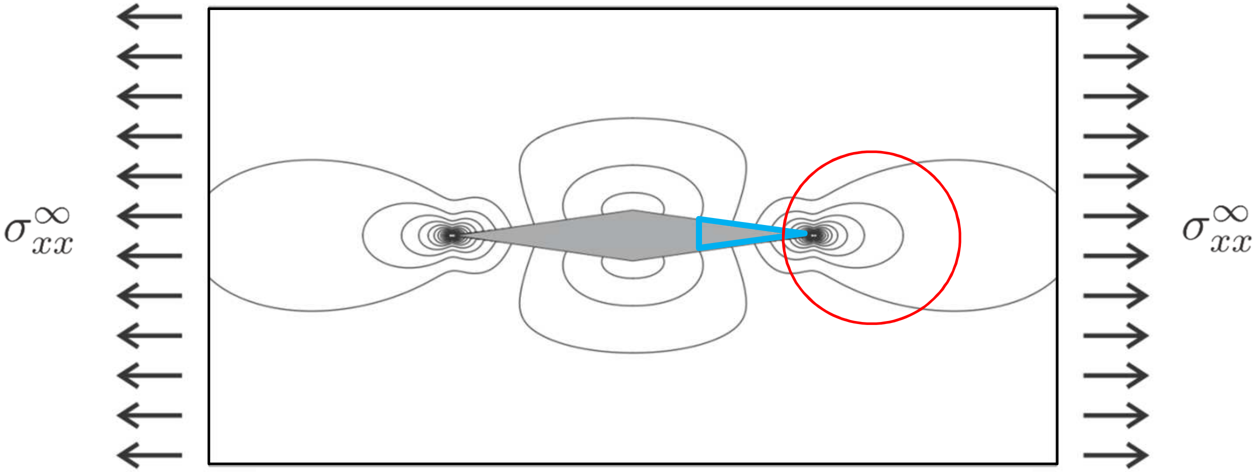
$$\psi^{(p)}(\zeta) = \left(8.1122 + 28.1115\zeta^2 + 1.8150\zeta^4 - 0.6928\zeta^6 + 0.4105\zeta^8 - 0.4451\zeta^{10} + 0.1665\zeta^{12} - 0.3417\zeta^{14} + 0.2727\zeta^{16}\right) R\,\sigma_{xx}^{\infty}\,\zeta / \left(-53.0727 + 44.1156\zeta^2 + 8.2012\zeta^4 - 2.2724\zeta^6 + 2.5225\zeta^8 - 1.3283\zeta^{10} + 1.4453\zeta^{12} - 0.9307\zeta^{14} + \zeta^{16}\right).$$

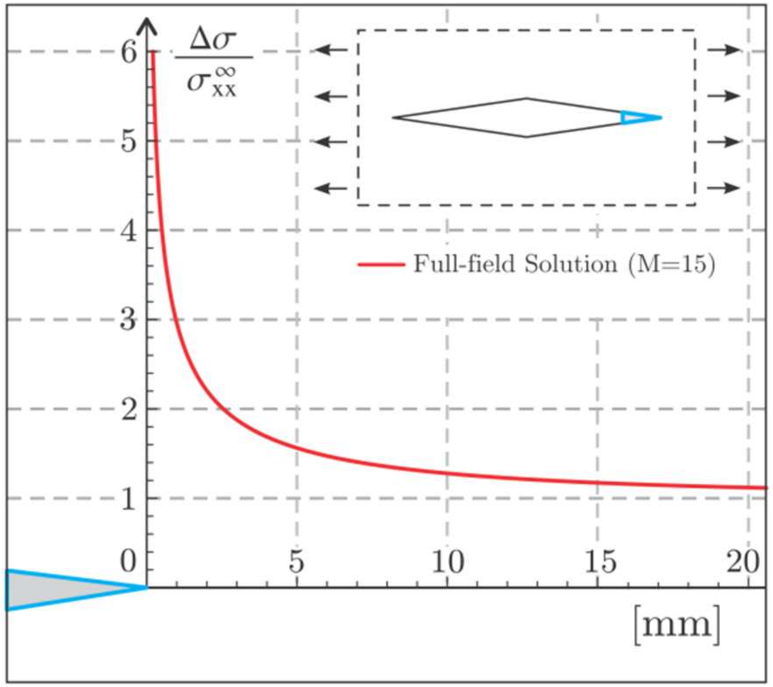
Stress field in transformed plane can be obtained as

$$\begin{cases} \sigma_{xx} + \sigma_{yy} = 4\mathrm{Re}\left[\dfrac{\varphi'(\zeta)}{\omega'(\zeta)}\right], \\ \sigma_{yy} - \sigma_{xx} + 2\,\mathrm{i}\,\sigma_{xy} = 2\left[\dfrac{\psi'(\zeta)}{\omega'(\zeta)} + \dfrac{\overline{\omega(\zeta)}}{\omega'(\zeta)^3}\left[\varphi''(\zeta)\omega'(\zeta) - \varphi'(\zeta)\omega''(\zeta)\right]\right] \\ 2\mu(u_x + \mathrm{i}\,u_y) = \kappa\varphi(\zeta) - \dfrac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi'(\zeta)} - \overline{\psi(\zeta)}. \end{cases}$$

IN-PLANE PROBLEM [5]

In-plane principal stress difference





OPEN QUESTIONS

- Are the singularities predicted in elasticity reliable ?

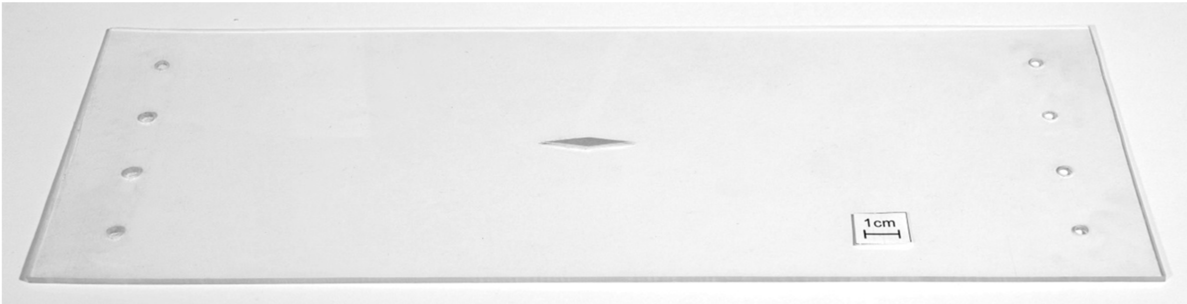
A ***real inclusion*** *has* ***finite stiffness*** *and*
its ***adhesion*** *with the matrix is necessarily* ***imperfect****…*

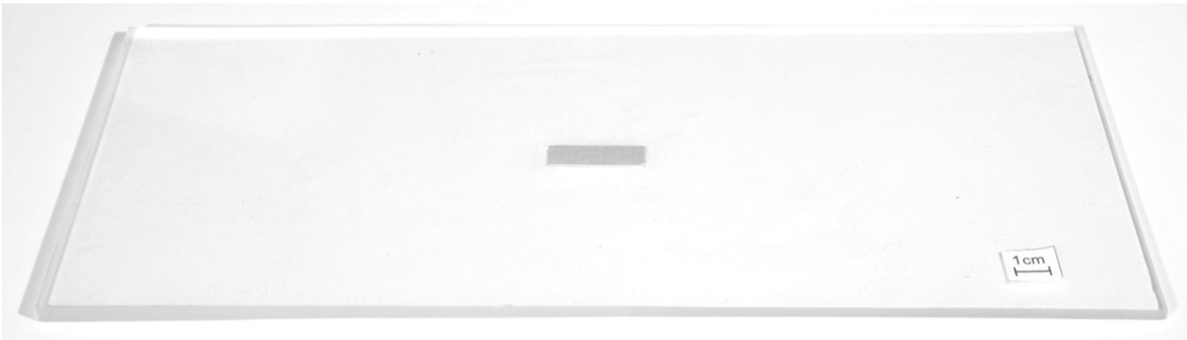
- Despite of the [strong] assumptions, is the rigid inclusion a sound model ?

EXPERIMENTAL WAY TO ANSWER

Matrix: transparent two-part epoxy resin

Polygonal inclusion: solid polycarbonate (3 mm thickness) with improved superficial rugosity

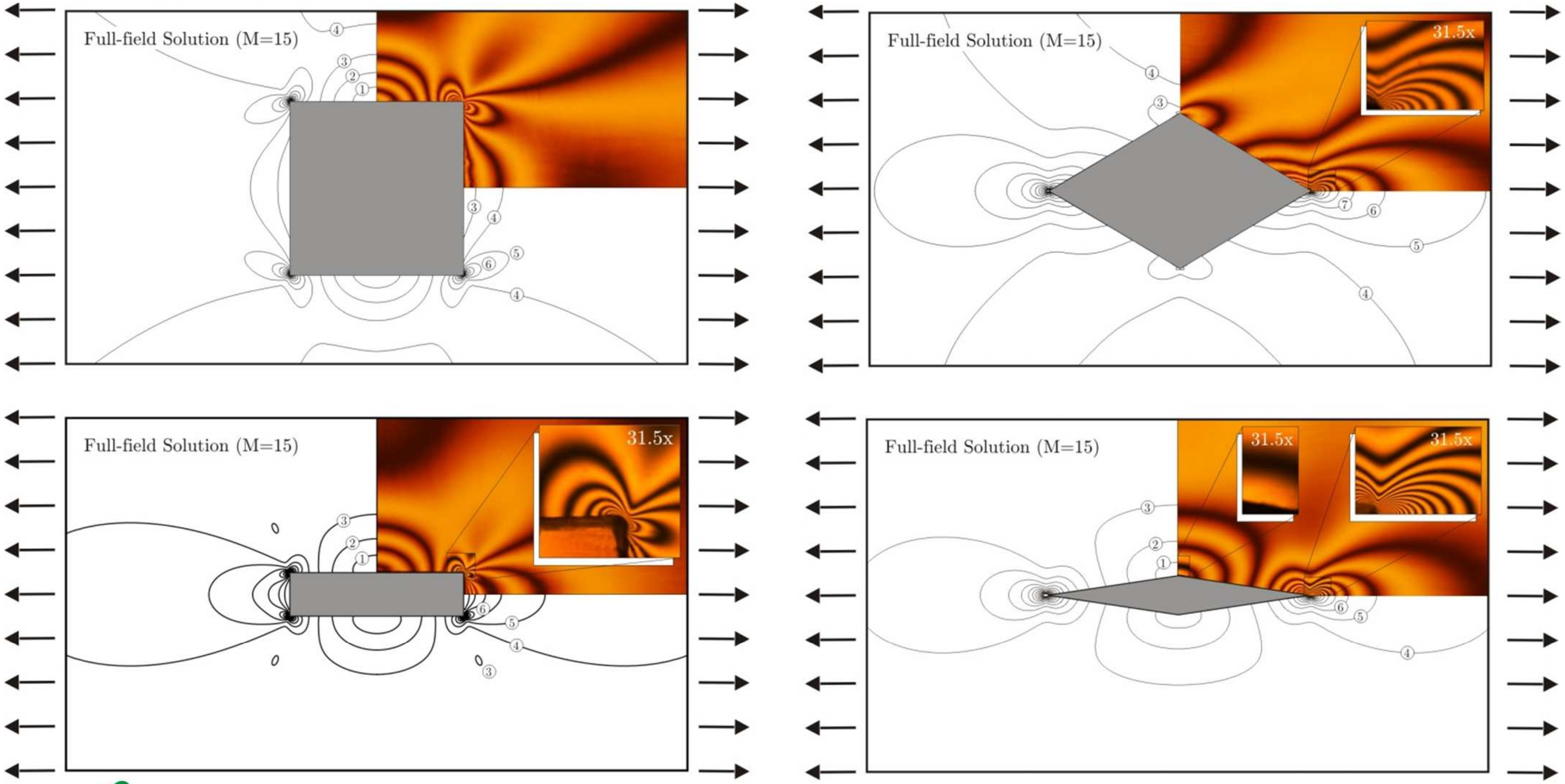




By performing photoelastic experiments on these samples

QUADRILATERAL STIFF INCLUSIONS

[Analytical solution vs photoelastic results]

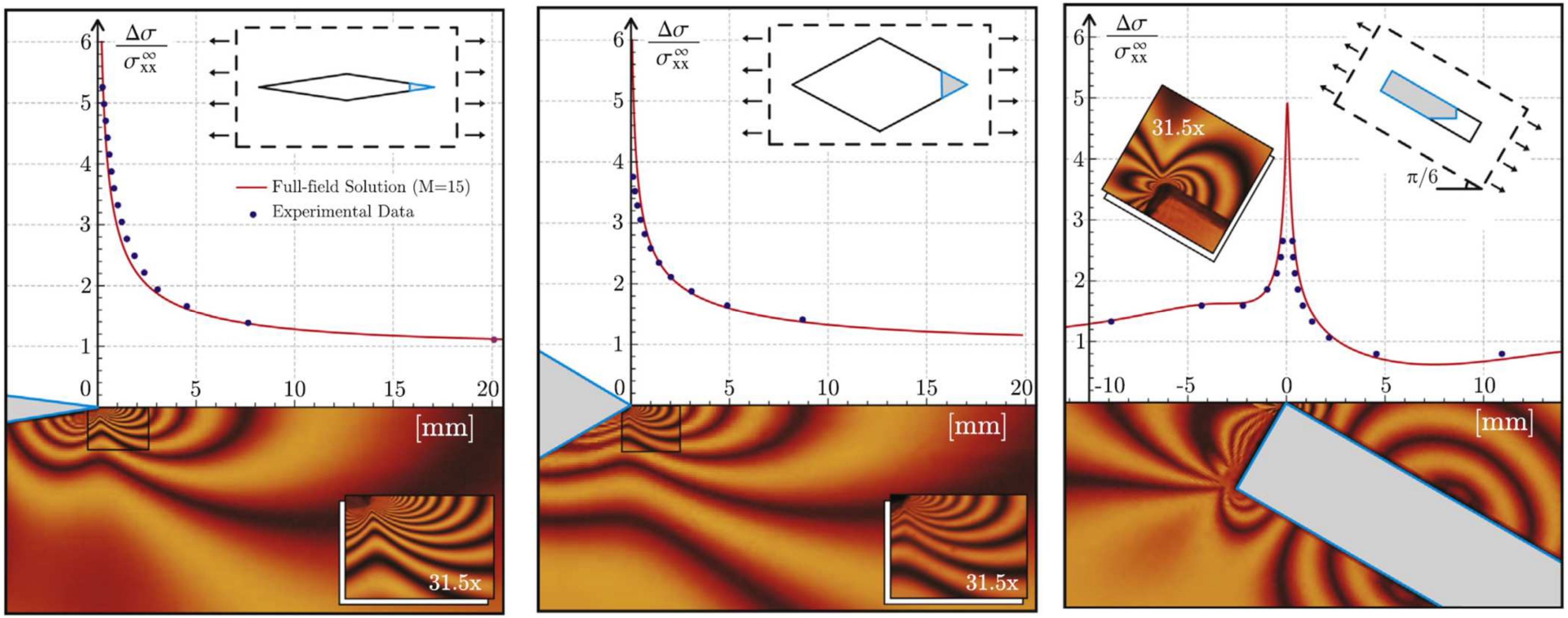


… THE POLYGONAL RIGID INCLUSION IS A SOUND MODEL!!

Misseroni, D. Dal Corso F. Shahzad S. and Bigoni, D., 2014. ***Engineering Fracture Mechanics 121-122, 87-97*****.**

A stress magnification factors of … are observed!!

5.3 3.8 2.7



Misseroni, D. Dal Corso F. Shahzad S. and Bigoni, D., 2014. *Engineering Fracture Mechanics 121-122, 87-97*.

FURTHER RESEARCH GOALS

- Can an inclusion be invisible to stress field ?
- Can the stress be annihilated by the presence of the inclusion ?

OUT-OF-PLANE PROBLEM & NONUNIFORM REMOTE SHEAR[1]

The out-of-plane problem in linear elasticity is governed by the Laplace's equation:

$$\nabla^2 w\left(x_1, x_2\right) = 0$$

The solution in terms of stress and displacement is given through a single complex potential:

$$w = \frac{1}{\mu}\mathrm{Re}[f(z)], \qquad \tau_{13} - \mathrm{i}\tau_{23} = f'(z)$$

Infinite class of nonuniform m-th order polynomial antiplane shear loads:

$$\tau_{13}^{\infty(m)}(x_1, x_2) = \sum_{j=0}^{m} b_j^{(m)} x_1^{m-j} x_2^j, \qquad \tau_{23}^{\infty(m)}(x_1, x_2) = \sum_{j=0}^{m} c_j^{(m)} x_1^{m-j} x_2^j$$

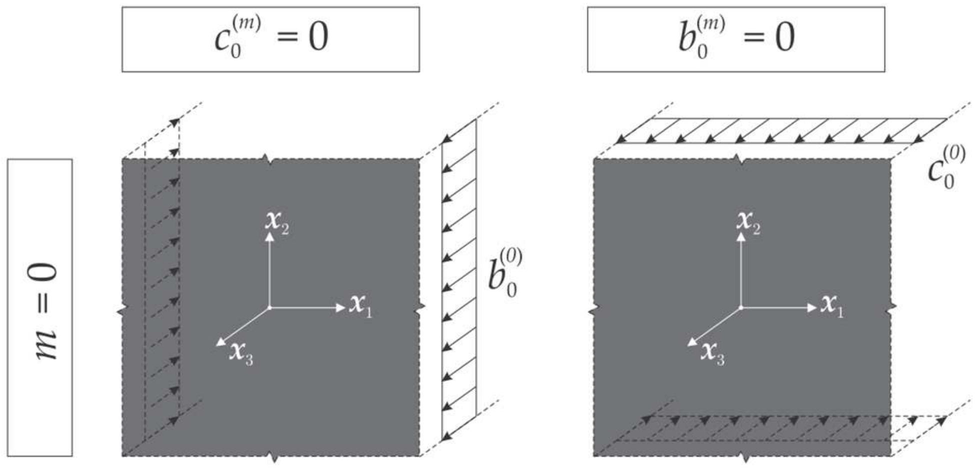
Dependent only on two **loading coefficients**…

$$b_j^{(m)} = \frac{(-1)^{\lfloor j/2 \rfloor}}{2}\binom{m}{j}\left[\boxed{b_0^{(m)}}(1+(-1)^j) + \boxed{c_0^{(m)}}(1-(-1)^j)\right]$$

$$c_j^{(m)} = \frac{(-1)^{\lceil j/2 \rceil}}{2}\binom{m}{j}\left[\boxed{b_0^{(m)}}(1-(-1)^j) + \boxed{c_0^{(m)}}(1+(-1)^j)\right]$$

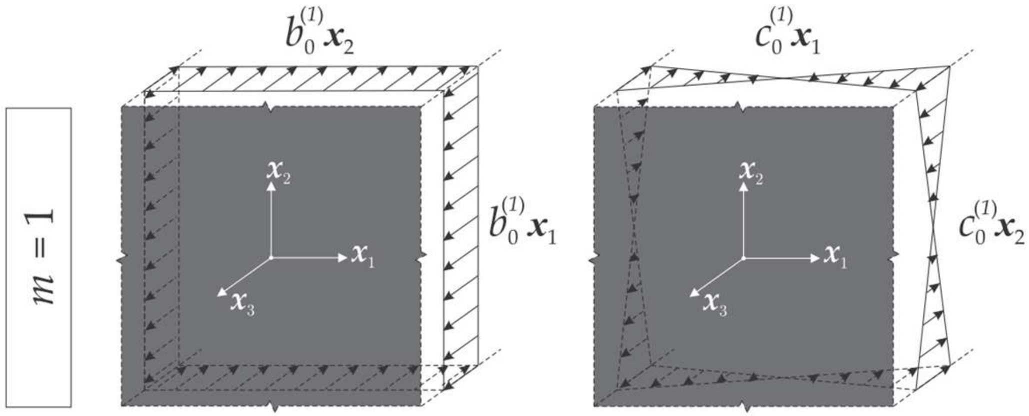
Uniform antiplane shear

$$\begin{cases} \tau_{13}^{\infty(0)}(x_1, 0) = b_0^{(0)}, \\ \tau_{23}^{\infty(0)}(x_1, 0) = c_0^{(0)}. \end{cases}$$



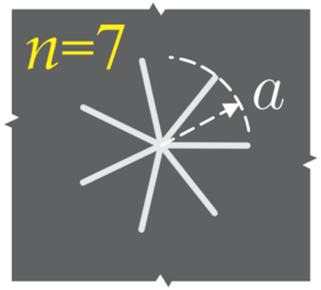
Linear antiplane shear

$$\begin{cases} \tau_{13}^{\infty(1)}(x_1, x_2) = b_0^{(1)} x_1 + c_0^{(1)} x_2, \\ \tau_{23}^{\infty(1)}(x_1, x_2) = c_0^{(1)} x_1 - b_0^{(1)} x_2. \end{cases}$$



STAR-SHAPED CRAKS OR STIFFENERS

The complex potential in ζ-plane can be expressed through the following relation



$$g(\zeta) = f(\omega(\zeta))$$

Further stress, displacement and resultant shear force in ζ-plane

$$w = \frac{1}{\mu}\mathrm{Re}[g(\zeta)], \qquad \tau_{13} - \mathrm{i}\tau_{23} = \frac{g'(\zeta)}{\omega'(\zeta)}, \qquad F_{\widehat{BC}} = \mathrm{Im}\left[g(\zeta_B) - g(\zeta_C)\right]$$

The **null traction** and **rigid displacement** boundary conditions:

$$F_{\widehat{BC}} = 0 \implies \text{void} \qquad\qquad w_B = w_C \implies \text{rigid}$$

Closed-form complex potential in ζ-plane via Generalized Binomial Theorem

$$\chi = 1 \quad \text{void}$$
$$\chi = -1 \quad \text{rigid}$$

$$g(\zeta, n, m) = \frac{a^{m+1}}{2^t}\left\{-\frac{\chi\overline{T^{(m)}}}{q!}\delta_{m+1,qn}\prod_{l=0}^{q-1}(t-l) + \sum_{j=0}^{q}\frac{1}{j!}\left(\prod_{l=0}^{j-1}(t-l)\right)\left[T^{(m)}\zeta^{m+1-jn} + \frac{\chi\overline{T^{(m)}}}{\zeta^{m+1-jn}}\right]\right\}$$

$$q = \left\lfloor \frac{m+1}{n} \right\rfloor \qquad t = \frac{2(m+1)}{n} \qquad T^{(m)} = \frac{b_0^{(m)} - \mathrm{i}\, c_0^{(m)}}{m+1}$$

ISOTOXAL STAR-SHAPED POLYGONAL INCLUSIONS

Through the Multinomial Theorem, the imposition of boundary conditions leads to the **analytical complex potential** in the conformal plane…

$$g(\zeta,\xi,n,m) = (a\Omega(n,\xi))^{m+1}\left[-\chi\overline{T^{(m)}}L_{m+1-qn}\,\delta_{m+1,qn} + \sum_{j=0}^{q} L_{m+1-jn}\left(T^{(m)}\zeta^{m+1-jn} + \frac{\chi\overline{T^{(m)}}}{\zeta^{m+1-jn}}\right)\right]$$

where coefficients

$$L_{m+1-jn} = \sum_{\mathcal{C}_j(l_0,l_1,..,l_\infty)} \binom{m+1}{l_0,l_1,\cdots,l_\infty} \prod_{k=0}^{\infty} (d_{1-kn})^{l_k}$$

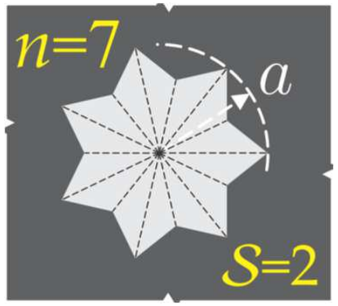
coefficients of the conformal map

The symbol in brackets is the multinomial coefficients

$$\binom{m+1}{l_0,l_1,\cdots,l_\infty} = \frac{(m+1)!}{l_0!\,l_1!\cdots l_\infty!}$$

with the sum must respect the following double condition

$$\mathcal{C}_j\,(l_0,l_1,..,l_\infty) : \left\{\sum_{k=0}^{\infty} l_k = m+1 \;\;\bigcap\;\; \sum_{k=1}^{\infty} k\,l_k = j\right\}$$



When m=0

UNIFORM LOADINGS

$$g(\zeta,\xi,n) = a\Omega(n,\xi)\left[T^{(0)}\zeta + \frac{\chi\overline{T^{(0)}}}{\zeta}\right]$$

Star-Shaped Cracks or Stiffeners

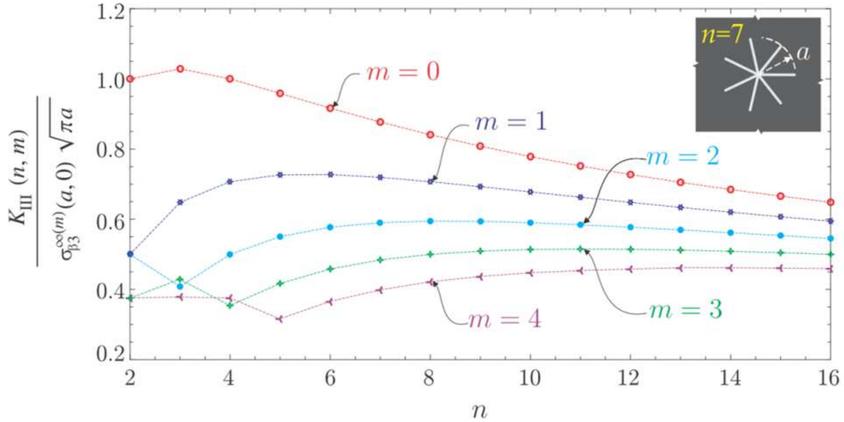
Sih, G.C., 1965. *J. Appl. Mech. 32, 51-58.*

Regular and Quadrilateral Polygons

Kohno Y. Ishikawa H., 1995. *Int. J. Eng. Sci. 33, 1547-1560.*

SIFs for **star-shaped cracks and stiffeners**:

$$\begin{bmatrix} K_{\mathrm{III}}^{\star\mathrm{S}}(n,m) \\ K_{\mathrm{III}}^{\star\mathrm{A}}(n,m) \end{bmatrix} = \frac{2^{\frac{3n-4(m+1)}{2n}}\, a^m}{m+1}\sqrt{\frac{\pi a}{n}} \times \left[\sum_{j=0}^{q} \frac{(m+1-jn)}{j!} \prod_{l=0}^{j-1}(t-l)\right] \begin{bmatrix} b_0^{(m)} \\ c_0^{(m)} \end{bmatrix}$$

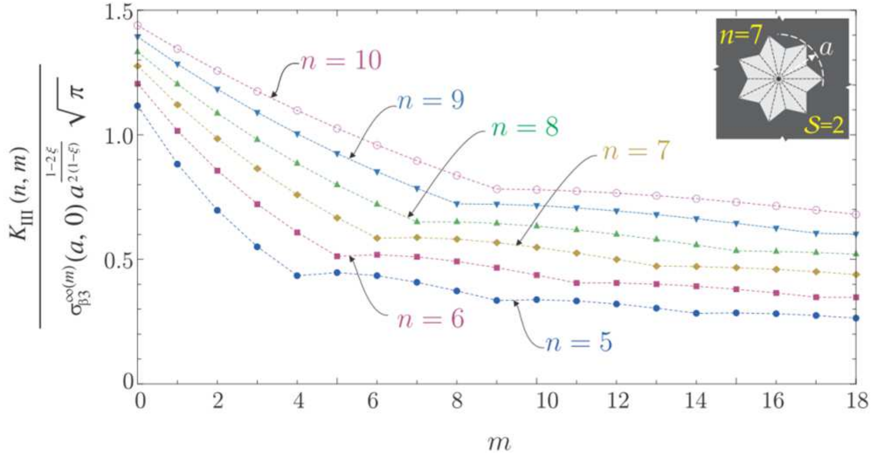


$m = 0$

Sih, G.C., 1965. *J. Appl. Mech. 32, 51-58.*

NSIFs for **isotoxal**

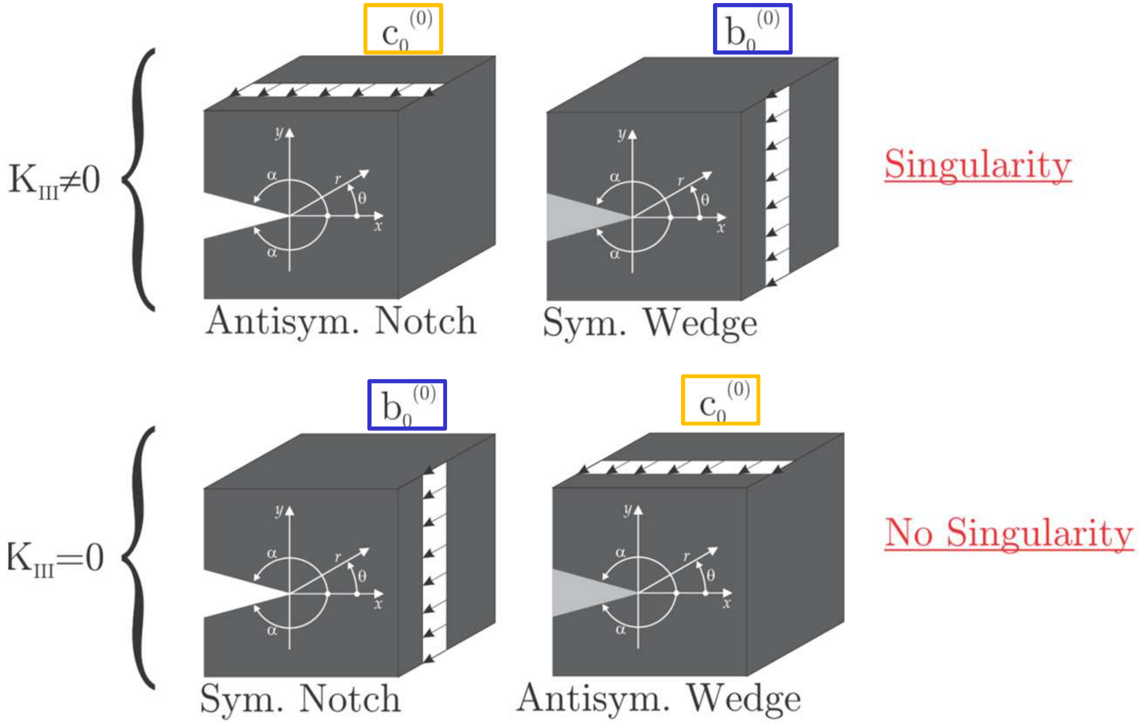
$$\begin{bmatrix} K_{\mathrm{III}}^{\star\mathrm{S}}(n,m) \\ K_{\mathrm{III}}^{\star\mathrm{A}}(n,m) \end{bmatrix} = \frac{2\sqrt{2\pi}\,(a\,\Omega(n,\xi))^m}{(m+1)2^{\frac{1}{n(1-\xi)}}} \left(\frac{a\,\Omega(n,\xi)}{n(1-\xi)}\right)^{\frac{1-2\xi}{2(1-\xi)}} \times \sum_{j=0}^{q}(m+1-jn)L_{m+1-jn} \begin{bmatrix} b_0^{(m)} \\ c_0^{(m)} \end{bmatrix}$$



Dal Corso F. Shahzad S. and Bigoni, D. 2016. Part.II. *Int. J. of Solids and Structures 85-86, 76-88.*

INVISIBILITY AND ANNIHILATION?

$$\begin{Bmatrix} K^{\star S}_{III}(n,m) \\ K^{\star A}_{III}(n,m) \end{Bmatrix} \sim \begin{Bmatrix} b_0^{(m)} \\ c_0^{(m)} \end{Bmatrix}$$



Stress annihilation and invisbility occurs at all vertices when

$$m = [3 - (-1)^n]\,\frac{j\,n}{4} - 1$$

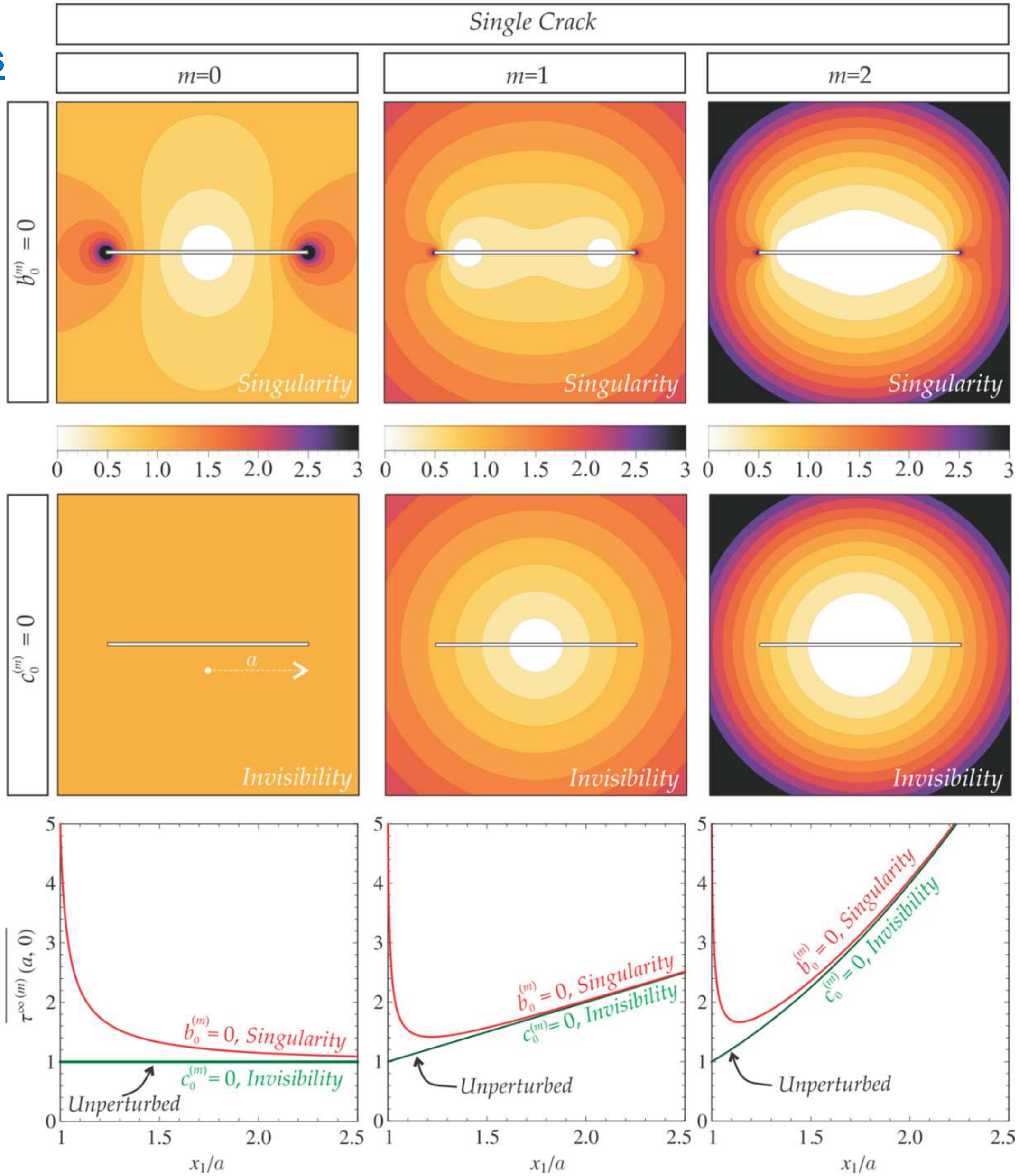
INVISIBILITY FOR SINGLE CRACK

Level sets of dimensionless shear stress

$$\frac{\tau^{(m)}(x_1,\, x_2)}{\tau^{\infty(m)}(a,\, 0)}$$

Shear stress

$$\tau = \sqrt{(\tau_{13})^2 + (\tau_{23})^2}.$$



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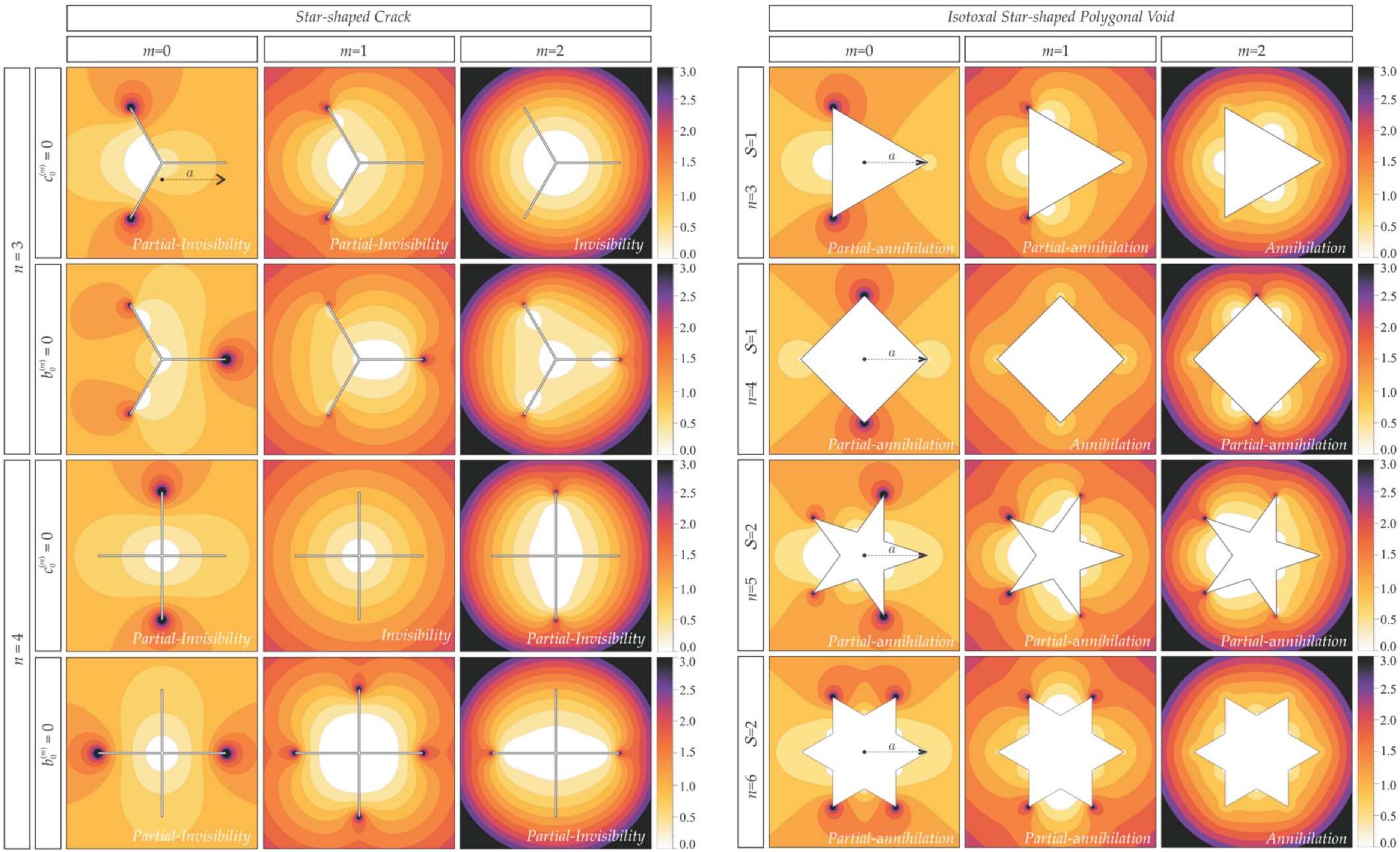
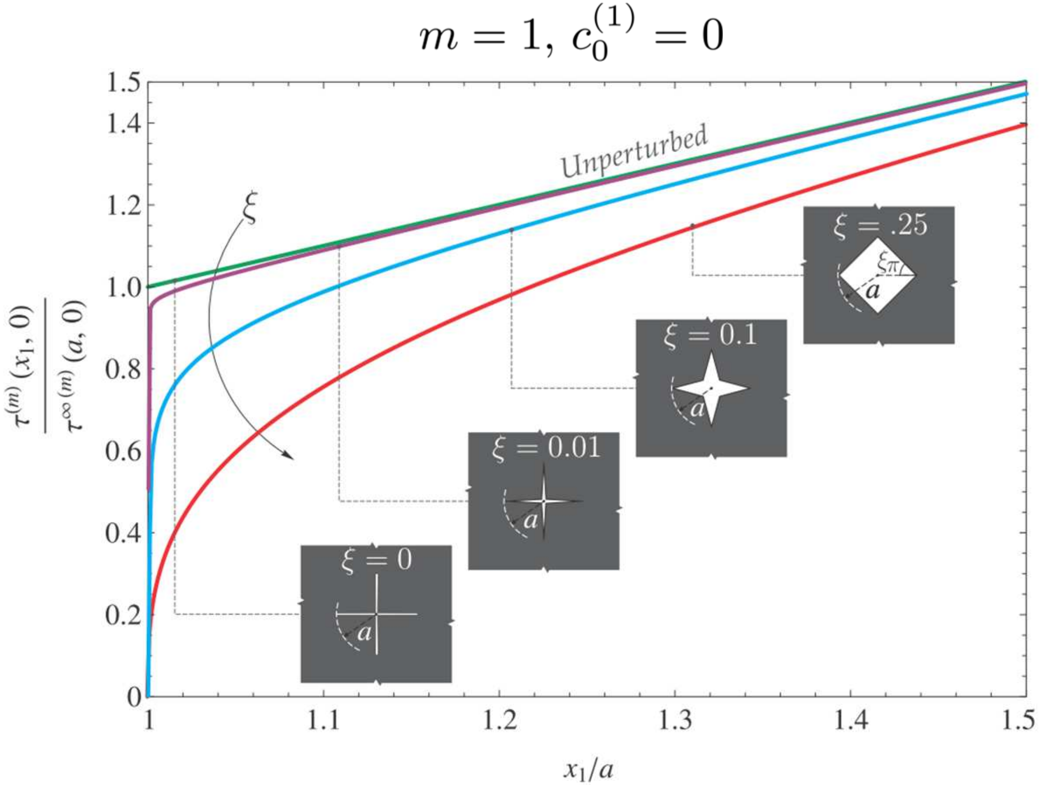
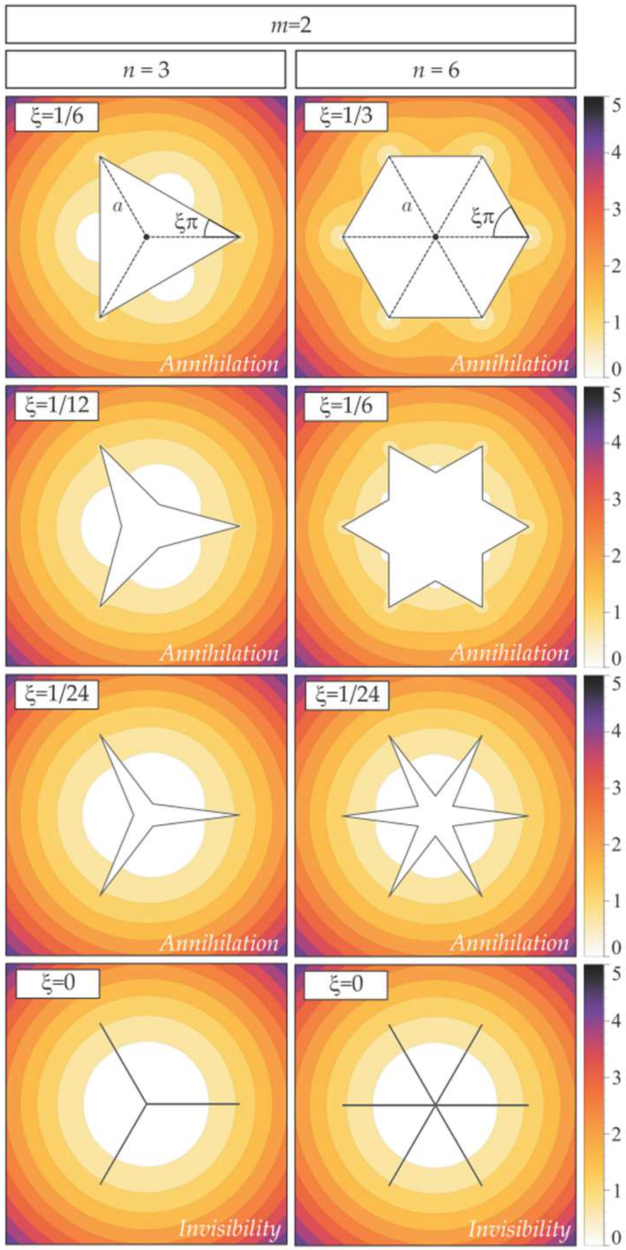
INVISIBILITY FOR STAR-SHAPED CRACK



Dal Corso F. Shahzad S. and Bigoni, D. 2016. Part.I. *Int. J. of Solids and Structures 85-86, 67-75.*

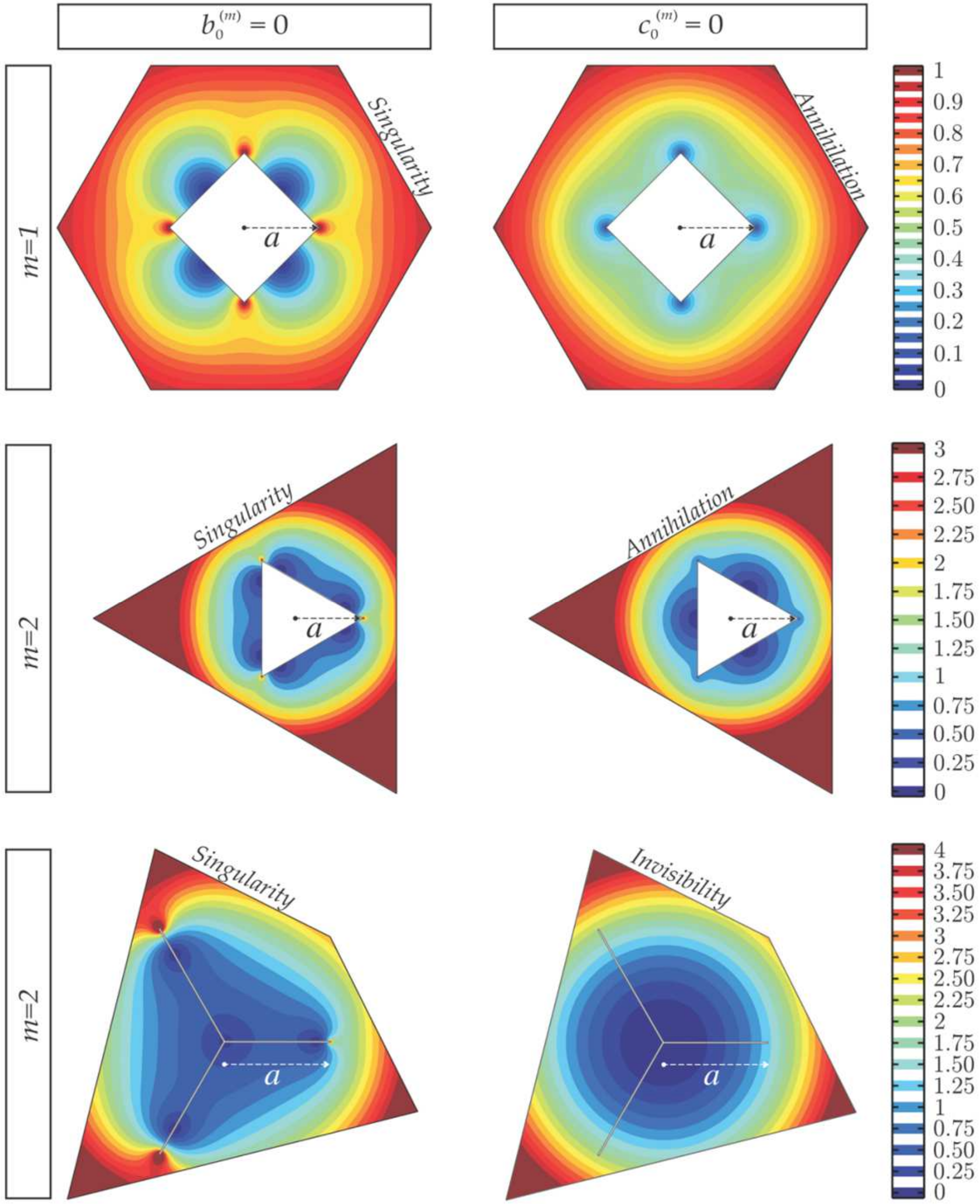
TRANSITION FROM ISOTOXAL TO STAR-SHAPED INLCUSION





Dal Corso F. Shahzad S. and Bigoni, D. 2016. Part.II. *Int. J. of Solids and Structures 85-86, 76-88.*

BEYOND THE HYPOTHESES OF INFINITE & REGULAR DOMAIN



Dal Corso F. Shahzad S. and Bigoni, D. 2016. Part.II. *Int. J. of Solids and Structures 85-86, 76-88.*

HYPOCYCLOIDAL-SHAPED INCLUSIONS[1]

- Can the stress at an inclusion vertex be smaller than that obtained when the inclusion is absent

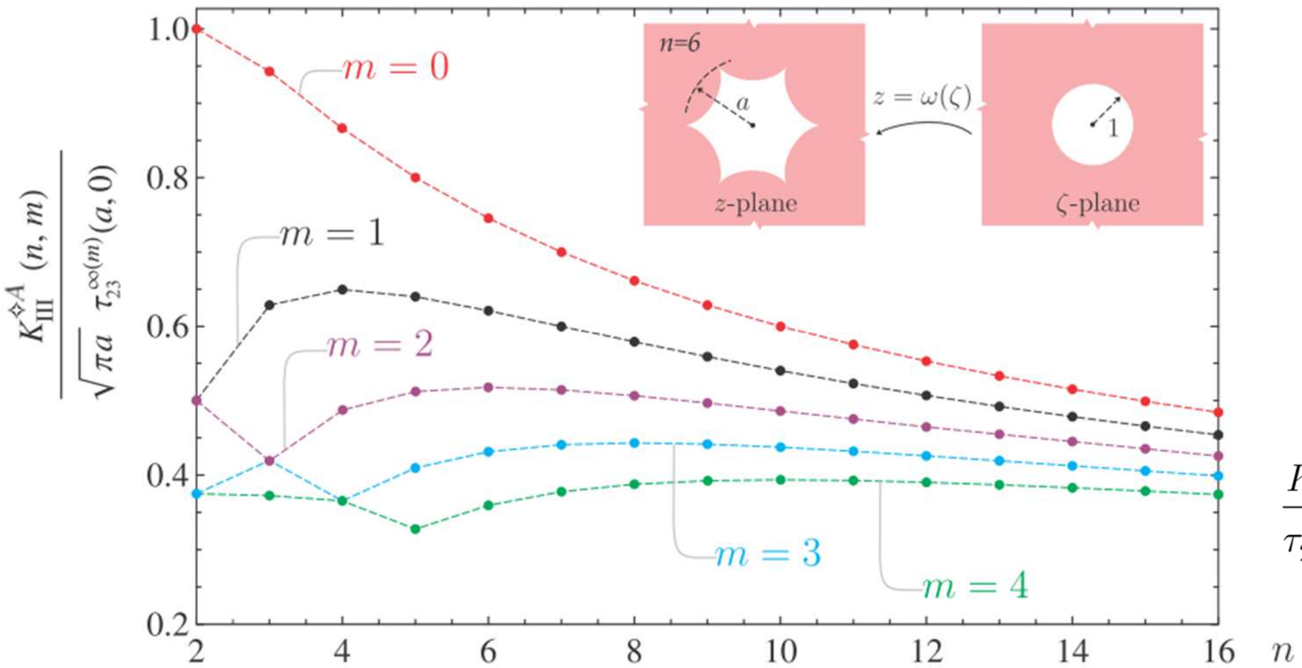
HYPOCYCLOIDAL-SHAPED INCLUSIONS[2]

Analytical solution

$$g(\zeta) = (a\Omega(n))^{m+1}\left[-\frac{(qn)!}{q!\,[q(n-1)]!\,(n-1)^q}\chi\overline{T^{(m)}}\delta_{m+1,qn} + \sum_{j=0}^{q}\binom{m+1}{j}\frac{1}{(n-1)^j}\left(T^{(m)}\zeta^{m+1-jn} + \frac{\chi\overline{T^{(m)}}}{\zeta^{m+1-jn}}\right)\right]$$

Stress Intensity Factors

$$\begin{bmatrix} K_{\mathrm{III}}^{\mathrm{S}}(n,m) \\ K_{\mathrm{III}}^{\mathrm{A}}(n,m) \end{bmatrix} = \frac{\sqrt{\pi a}(n-1)^{m+\frac{1}{2}}a^m}{n^{m+1}(m+1)}\left[\sum_{j=0}^{q}\binom{m+1}{j}\frac{(m+1-jn)}{(n-1)^j}\right]\begin{bmatrix}(1-\chi)b_0^{(m)} \\ (1+\chi)c_0^{(m)}\end{bmatrix}$$



$m = 0$

Chen, Y.Z., Hasebe, N., and Lee, K.Y., 2003.

$$\frac{K_{\mathrm{III}}^{\diamond\mathrm{A}}(n,m)}{\tau_{23}^{\infty(m)}(a,0)} \geq \frac{K_{\mathrm{III}}^{\diamond\mathrm{A}}(n,m+1)}{\tau_{23}^{\infty(m+1)}(a,0)}$$

Shahzad S., Dal Corso F. and Bigoni, D. 2016. *Submitted.*

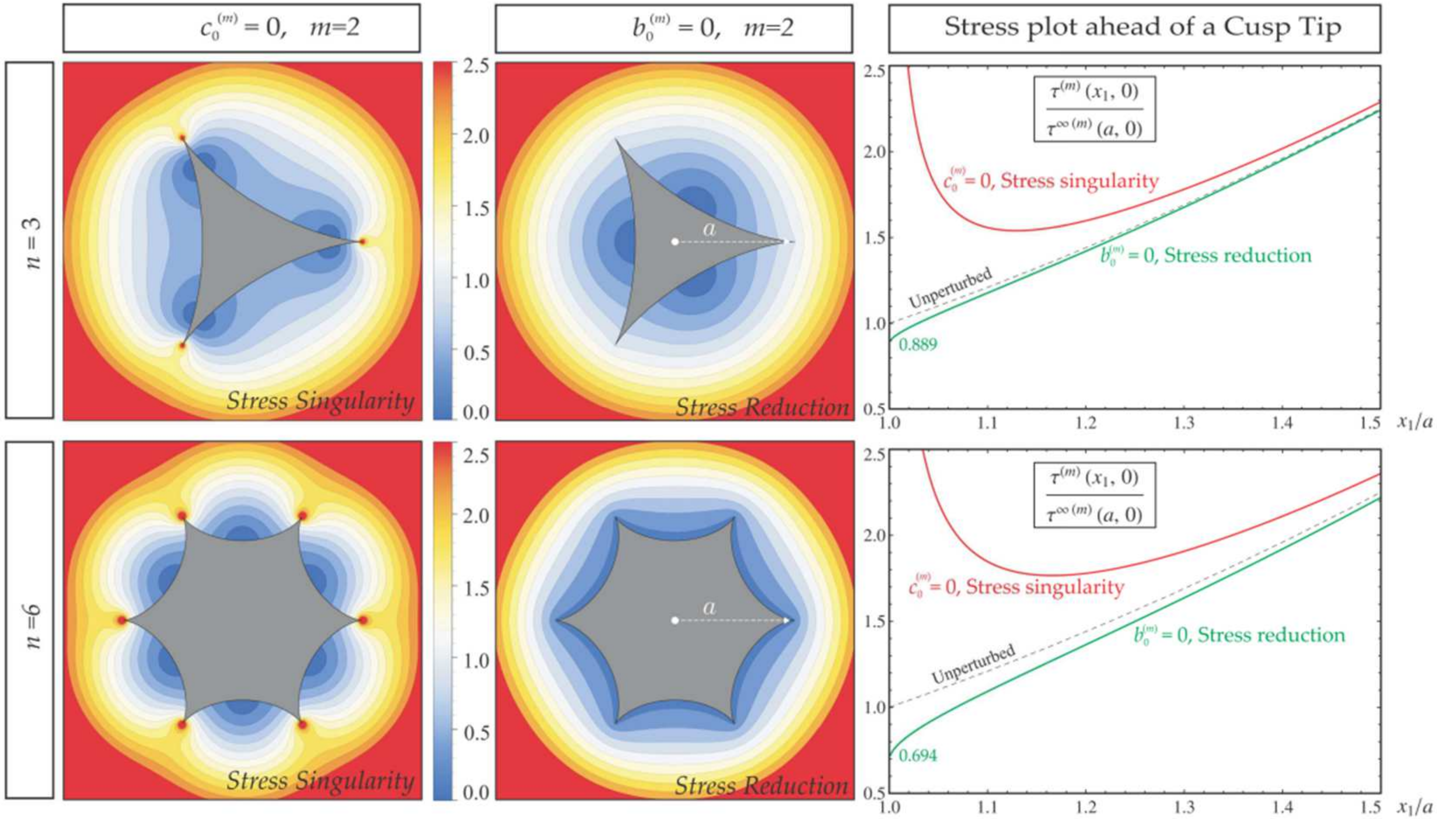
HYPOCYCLOIDAL-SHAPED INCLUSIONS[3]

Stress Reduction Factors

$$\mathrm{SRF}(n,m) := 1 - \frac{\tau^{(m)}(a,0)}{\tau^{\infty(m)}(a,0)} = 1 - \mathcal{A}(n,m) \in [0;1)$$

$$\mathcal{A}(n,m) = \frac{2}{(m+1)n^{m+1}} \sum_{j=0}^{q} \binom{m+1}{j} \frac{(m+1-jn)^2}{(n-1)^{j-m}}$$

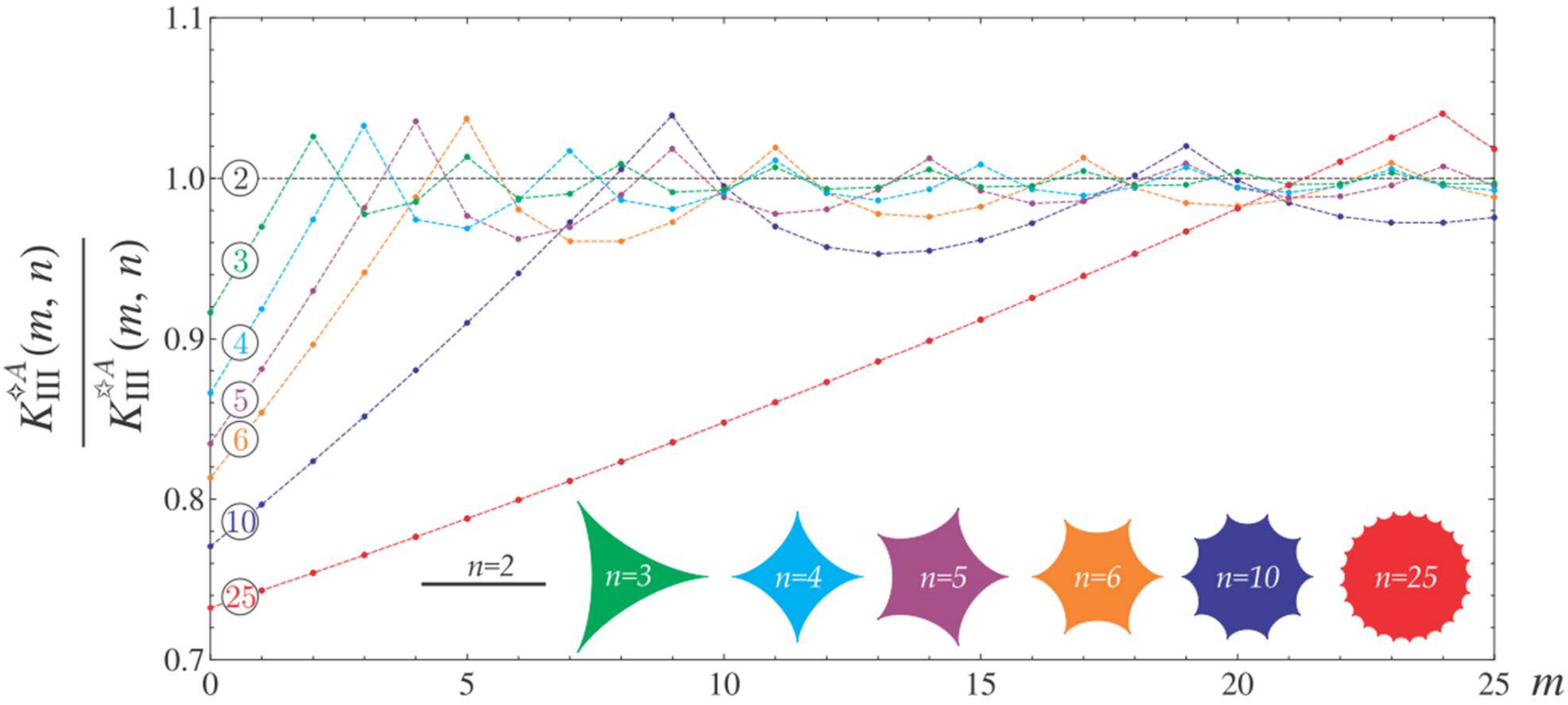
Stress reduction occurs at all cusps when $m = [3-(-1)^n]\,\frac{j\,n}{4} - 1$



Shahzad S., Dal Corso F. and Bigoni, D. 2016. *Submitted.*

HYPOCYCLOIDAL-SHAPED INCLUSIONS[4]

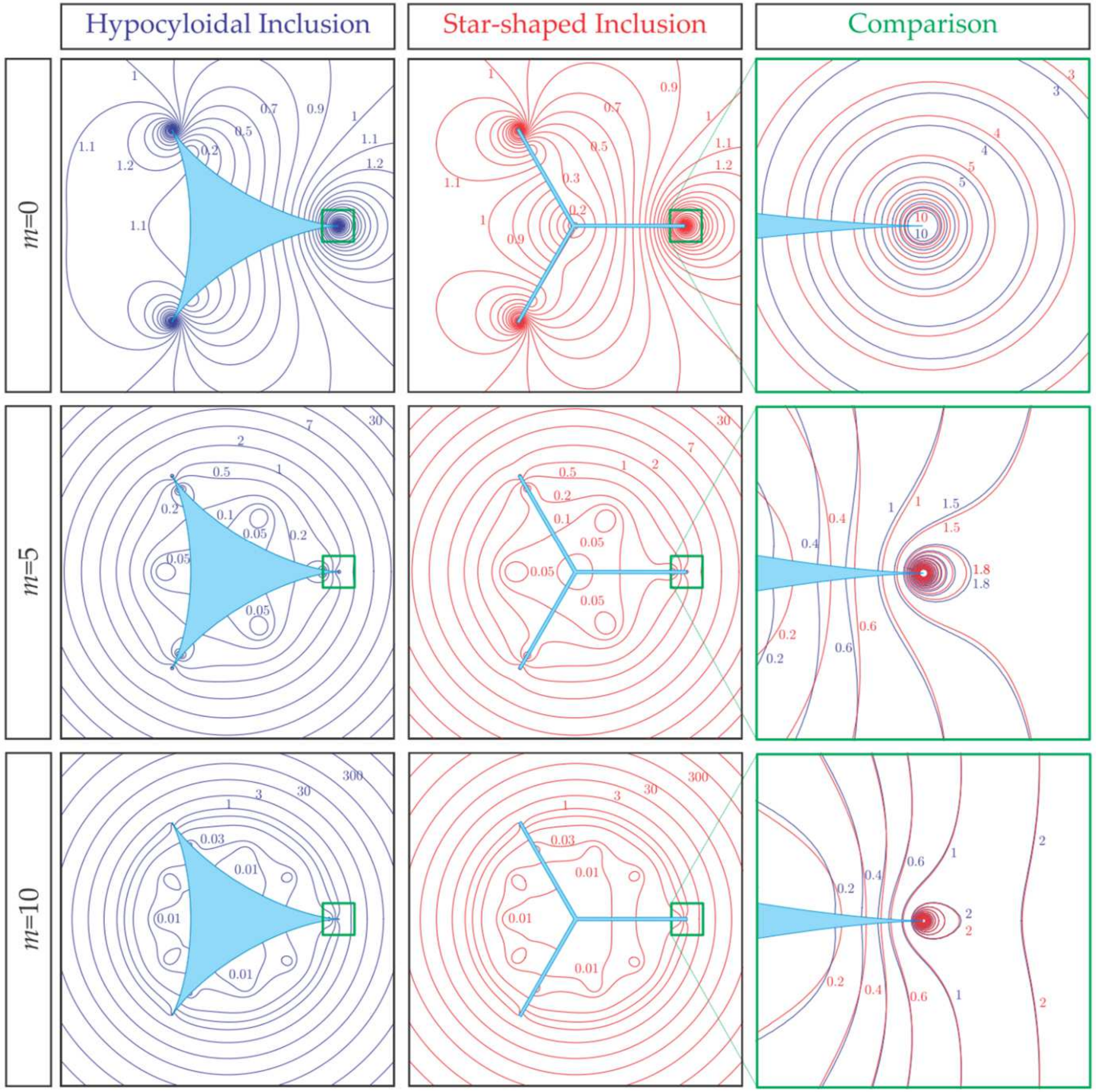
Equivalence between star-shaped cracks and hypocycloidal-shaped voids



Shahzad S., Dal Corso F. and Bigoni, D. 2016. *Submitted.*

HYPOCYCLOIDAL-SHAPED INCLUSIONS[5]

Equivalence between star-shaped cracks and hypocycloidal-shaped rigid inclusions



Shahzad S., Dal Corso F. and Bigoni, D. 2016. *Submitted.*

Airy Stress Function for infinite class of polynomial fields

$$F^{(\infty)}(r,\theta,m)=r^{m+2}\Big[B^{(m)}\cos(m\theta)-C^{(m)}\sin(m\theta)+D^{(m)}\cos\left((m+2)\theta\right)-E^{(m)}\sin\left((m+2)\theta\right)\Big]$$

Unperturbed complex potentials

$$\begin{cases}\varphi^{\infty}(z,m) &= \overbrace{\left(B^{(m)}+\mathrm{i}C^{(m)}\right)}^{\Gamma^{(m)}}\,z^{m+1},\\ \psi^{\infty}(z,m) &= \underbrace{(m+2)\left(D^{(m)}+\mathrm{i}E^{(m)}\right)}_{\Gamma'^{(m)}}\,z^{m+1},\end{cases}$$

$$m=0 \;\Rightarrow\; \begin{cases}B^{(0)}=\dfrac{\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}}{4}\\ C^{(0)}=\dfrac{2\mu\varpi^{\infty}}{1+\kappa}\\ D^{(0)}=\dfrac{\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}}{4}\\ E^{(0)}=\dfrac{\tau_{xy}^{\infty}}{2}\end{cases}$$

Elliptical void and rigid inclusion

$$\omega(\zeta)=R\left(\frac{1}{\zeta}+S\,\zeta\right)$$

$$R=\frac{a+b}{2}>0;\qquad\qquad 0\leq S=\frac{a-b}{a+b}\leq 1;$$

Complex potential

$$
\begin{aligned}
\varphi(\zeta) = {} & \overbrace{R^t\,\Gamma \sum_{j=q+1}^{t} \binom{t}{j} \frac{S^{t-j}}{\zeta^{2j-t}}}^{\varphi^\infty(\zeta)} \\
& - \frac{R^t \chi\,\overline{\Gamma'}}{\Theta} \sum_{j=q+1}^{t} \binom{t}{j} S^{t-j} \zeta^{2j-t} \\
& + \frac{R^t \chi\,\overline{\Gamma}\,S}{\Theta} \sum_{j=q+1}^{t} \binom{t}{j} S^{t-j} (t-2j) \zeta^{2j-t} \\
& - \frac{R^t \chi\,\overline{\Gamma}(S^2+1)t}{\Theta} \begin{cases} \displaystyle\sum_{j=q+1}^{m} \binom{m}{j} S^{m-j} \zeta^{2j-t} & \text{if} \quad m \text{ is even} \\ \displaystyle\sum_{j=q}^{m} \binom{m}{j} S^{m-j} \zeta^{2j-t} & \text{if} \quad m \text{ is odd} \end{cases}
\end{aligned}
$$

Limit case of crack $S = 1$

$$
K = -2\sqrt{2\pi} \lim_{\zeta \to \zeta_0} \left[\sqrt{\omega(\zeta) - \omega(\zeta_0)}\, \frac{\varphi'(\zeta)}{\omega'(\zeta)} \right]
$$

$$
K = K_{\mathrm{I}} - \mathrm{i} K_{\mathrm{II}}
$$

STRESS ANNIHILATION, REDUCTION AND INVISIBILITY IN IN-PLANE PROB.

SIFs for crack

$$\left\{ \begin{array}{c} K_{\mathrm{I}}^{\star}(m) \\ K_{\mathrm{II}}^{\star}(m) \end{array} \right\} = {}_2F_1\left(\frac{1}{2}, -m; 2; 2\right) \left\{ \begin{array}{c} \widetilde{B}^{(m)} + \widetilde{D}^{(m)} \\ \widetilde{E}^{(m)} + m\widetilde{C}^{(m)} \end{array} \right\} a^{\frac{2m+1}{2}} \sqrt{\pi}.$$

$$\begin{cases} \widetilde{B}^{(m)} = (m+1)(m+2)B^{(m)} \\ \widetilde{D}^{(m)} = (m+1)(m+2)D^{(m)} \\ \widetilde{E}^{(m)} = (m+1)(m+2)E^{(m)} \\ \widetilde{C}^{(m)} = (m+1)C^{(m)}. \end{cases}$$

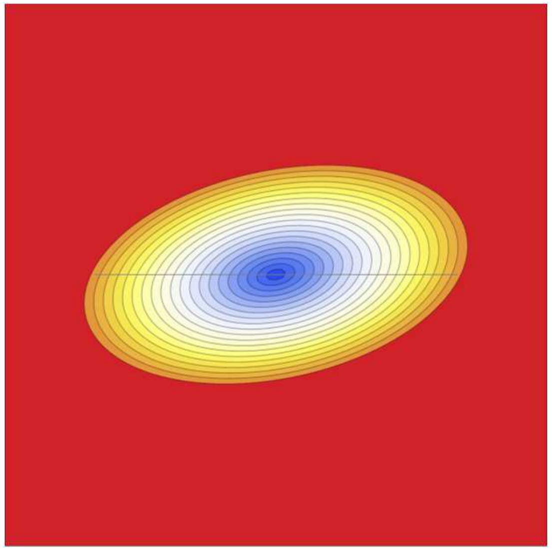
Invisibility conditions for **crack**

$$\tilde{D}^{(m)} = -\tilde{B}^{(m)}, \qquad \tilde{E}^{(m)} = -m\tilde{C}^{(m)}$$

SIFs for **stiffener**

$$\left\{ \begin{array}{c} K_{\mathrm{I}}^{\bigstar}(m) \\ K_{\mathrm{II}}^{\bigstar}(m) \end{array} \right\} = {}_2F_1\left(\frac{1}{2}, -m; 2; 2\right) \frac{1}{\kappa} \left\{ \begin{array}{c} \alpha\widetilde{B}^{(m)} - \widetilde{D}^{(m)} \\ -\widetilde{E}^{(m)} - \beta\widetilde{C}^{(m)} \end{array} \right\} a^{\frac{2m+1}{2}} \sqrt{\pi}$$

$$\alpha = \frac{(k-m-1)}{m+2}, \qquad \beta = k+m+1$$



Invisibility conditions for stiffeners

$$D^{(m)} = \alpha B^{(m)}, \qquad E^{(m)} = -\frac{\beta}{(m+2)} C^{(m)}$$

Work in progress

CONCLUSIONS

- Rigid inclusion model is a sound model
- Special geometries and loading conditions have been found for out-of-plane problem to provide
  - Invisible Star-shaped cracks and stiffeners
  - Isotoxal inclusions acting as stress annihilators
  - Hypocycloidal inclusions acting as stress reducers
- Preliminary investigations show that the invisibility, stress annihilation and reduction features can be realized for in-plane elasticity, but not simultaneously at all the inclusion vertexes.
- These results are fundamental in the understanding of the failure mechanics of composites and can be used to enhance engineering design towards high-strength materials.

Scientific Publications

1. **Shahzad, S.**, Dal Corso, F., and Bigoni, D. (2016)
   Hypocycloid inclusions in nonuniform out-of-plane elasticity: stress singularity vs stress reduction.
   Submitted.

2. Dal Corso, F., **Shahzad, S.**, Bigoni, D. (2016)
   Isotoxal star-shaped polygonal voids and rigid inclusions in a nonuniform antiplane shear fields. I. Formulation and full-field solution.
   International Journal of Solids and Structures, doi:10.1016/j.ijsolstr.2016.01.027.

3. Dal Corso, F., **Shahzad, S.**, Bigoni, D. (2016)
   Isotoxal star-shaped polygonal voids and rigid inclusions in nonuniform antiplane shear fields. II. Stress singularities, stress annihilation and inclusion invisibility.
   International Journal of Solids and Structures, doi:10.1016/j.ijsolstr.2016.01.026.

4. Misseroni, D., Dal Corso, F., **Shahzad, S.**, Bigoni, D. (2014)
   stress concentration near stiff inclusions: validation of rigid inclusion model and boundary layers by means of photoelasticity.
   Engineering Fracture Mechanics, 121-122, 87-97.

…THANK YOU FOR YOUR ATTENTION!

Paris' fatigue Law

Fatigue crack growth model widely used in materials science and fracture mechanics

the crack length

crack growth rate: $$\frac{da}{dN} = C\Delta K^m$$

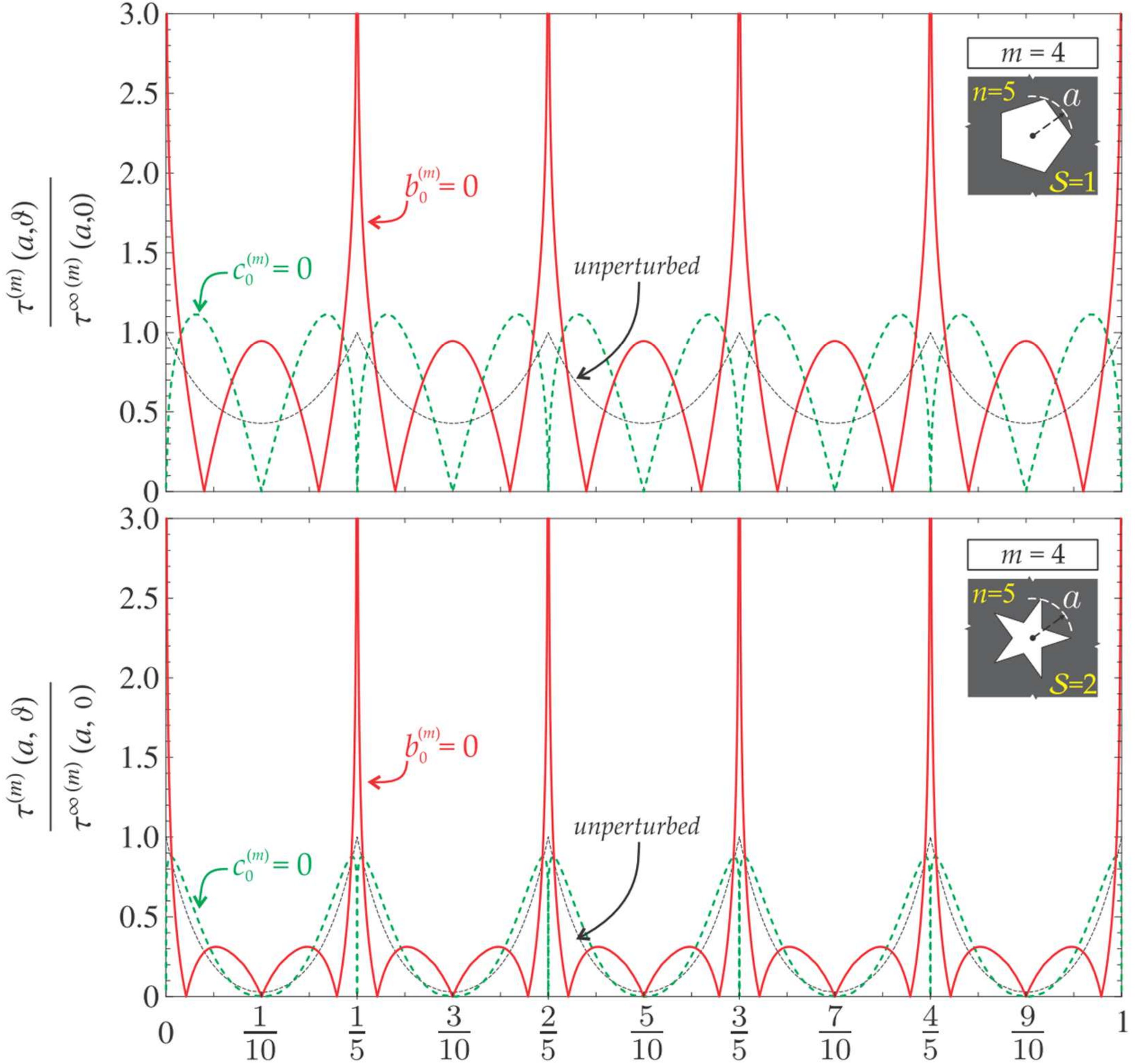
number of load cycles

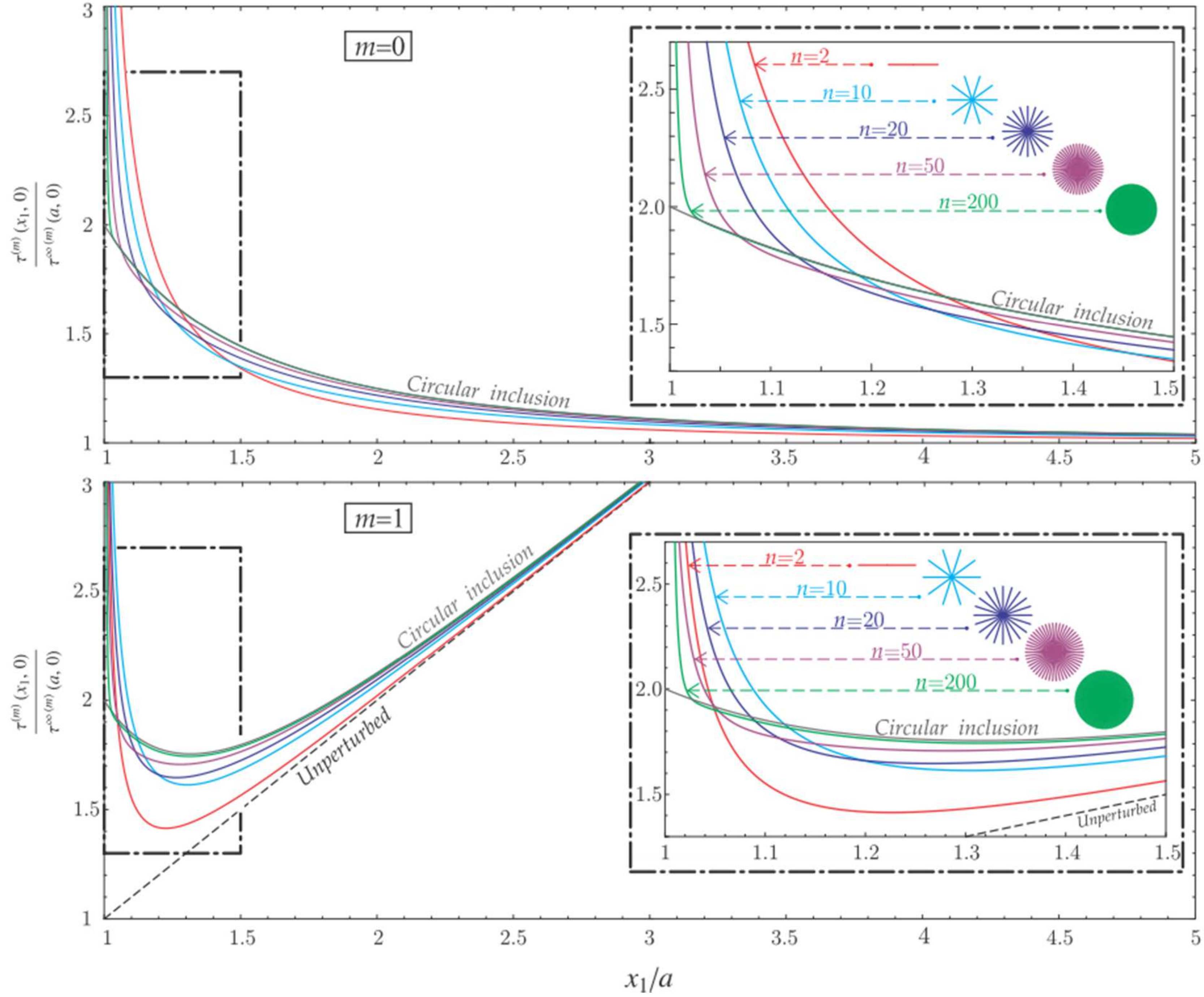
Range of the SIFs at max and min loading

$$\Delta K = K_{max} - K_{min}$$

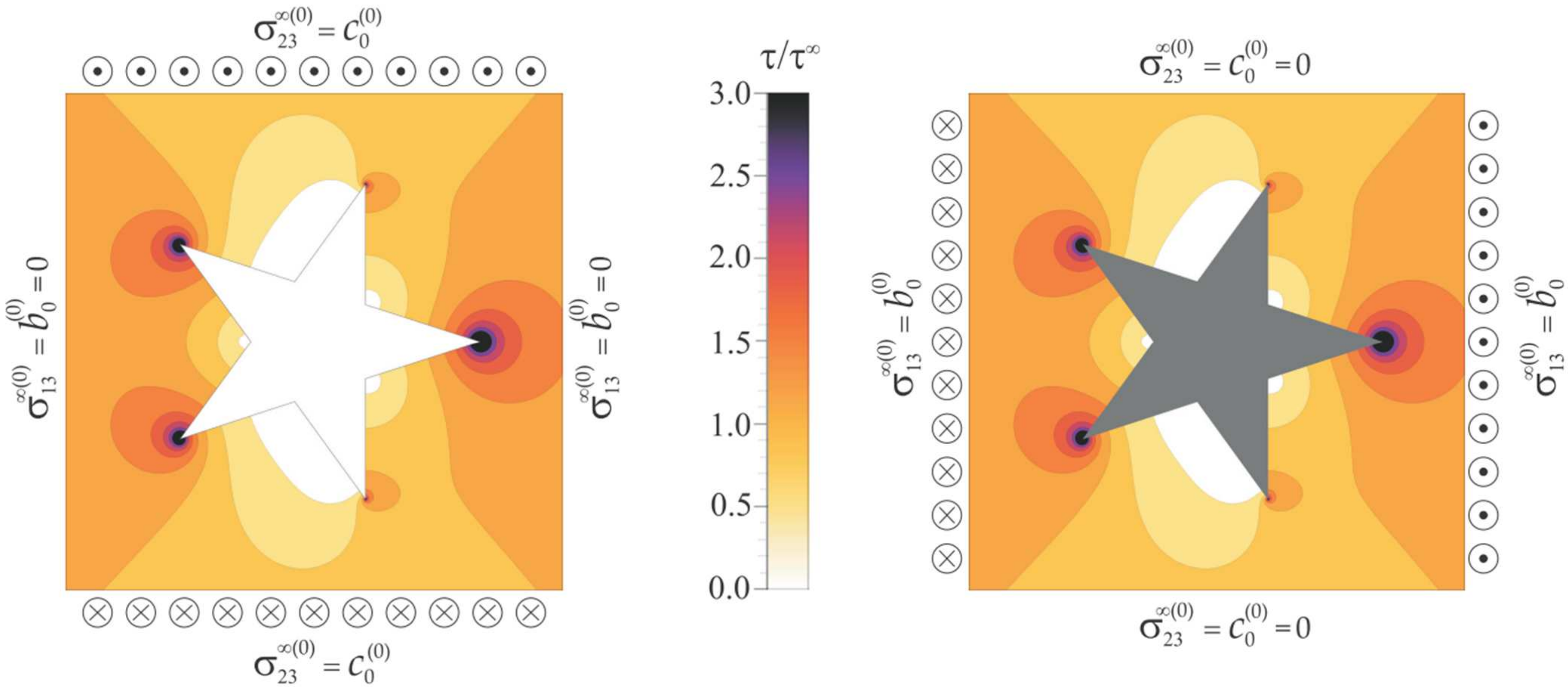
C and ***m*** are material constants

Used to quantify the residual life in terms of the remaining number of cycles to fracture!


m = 4
n=5
a
S=1
$b_0^{(m)} = 0$
$c_0^{(m)} = 0$
unperturbed
$\frac{\tau^{(m)}(a,\vartheta)}{\tau^{\infty(m)}(a,0)}$
m = 4
n=5
a
S=2
$b_0^{(m)} = 0$
$c_0^{(m)} = 0$
unperturbed
$\frac{\tau^{(m)}(a,\vartheta)}{\tau^{\infty(m)}(a,0)}$
0
0.5
1.0
1.5
2.0
2.5
3.0
0
$\frac{1}{10}$
$\frac{1}{5}$
$\frac{3}{10}$
$\frac{2}{5}$
$\frac{5}{10}$
$\frac{3}{5}$
$\frac{7}{10}$
$\frac{4}{5}$
$\frac{9}{10}$
1

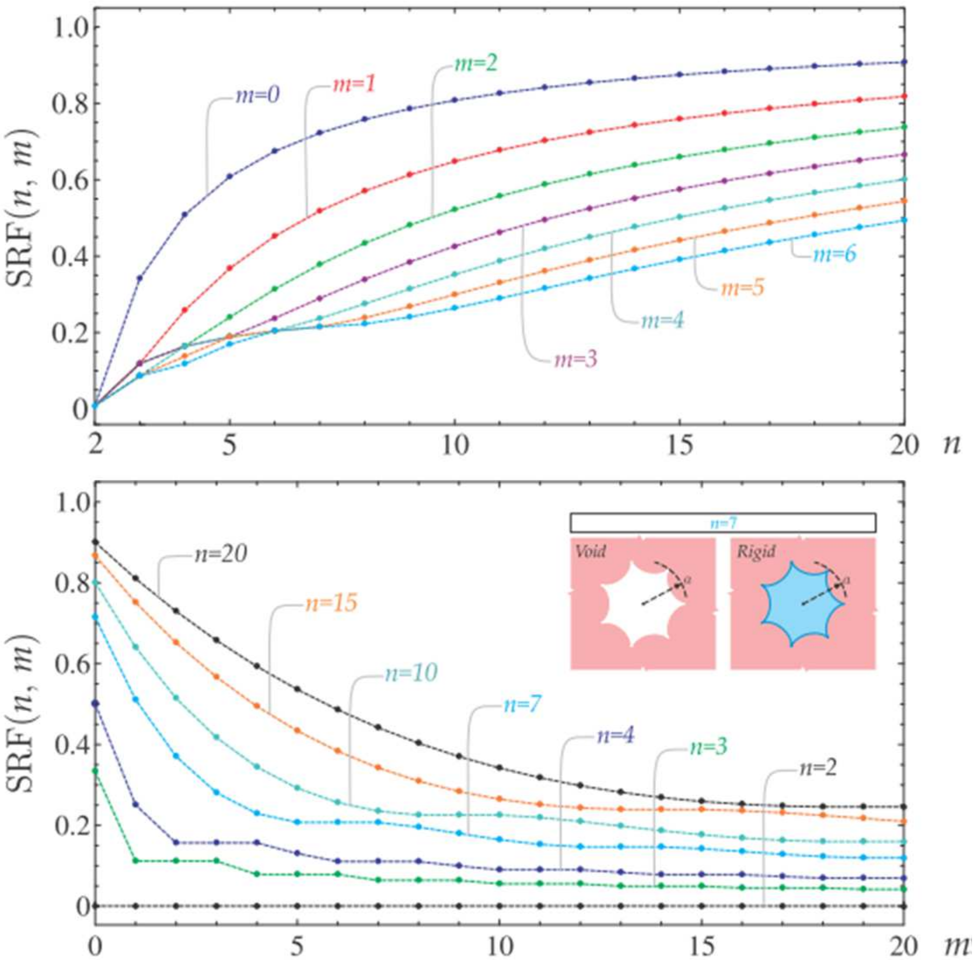


m=0
m=1
n=2
n=10
n=20
n=50
n=200
Circular inclusion
Unperturbed
$\frac{\tau^{(m)}(x_1, 0)}{\tau^{\infty(m)}(a, 0)}$
x_1/a



$\sigma_{23}^{\infty(0)} = c_0^{(0)}$
$\sigma_{13}^{\infty(0)} = b_0^{(0)} = 0$
$\sigma_{13}^{\infty(0)} = b_0^{(0)} = 0$
$\sigma_{23}^{\infty(0)} = c_0^{(0)}$
τ/τ^{∞}
3.0
2.5
2.0
1.5
1.0
0.5
0.0
$\sigma_{23}^{\infty(0)} = c_0^{(0)} = 0$
$\sigma_{13}^{\infty(0)} = b_0^{(0)}$
$\sigma_{13}^{\infty(0)} = b_0^{(0)}$
$\sigma_{23}^{\infty(0)} = c_0^{(0)} = 0$

Stress Reduction Factors



Some properties of SRFs are observed

i) $$\mathrm{SRF}(n, m = n) = \\ = \mathrm{SRF}(n, m = n-1) \\ = \mathrm{SRF}(n, m = n-2)$$

ii) $$\mathrm{SRF}(n, m+1) \leq \mathrm{SRF}(n, m) \leq \mathrm{SRF}(n+1, m)$$

Shahzad S., Dal Corso F. and Bigoni, D. 2016. *Submitted.*